Module 03
Linear Algebra Review & Solutions to State Space

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Vector Space (aka Linear Space)

A (real) vector space $V$ is a set with two operations:

- Vector sum $+: V + V \rightarrow V$
- Scalar multiplication $\cdot: \mathbb{R} \times V \rightarrow V$

that has the following properties:

1. **Commutative:** $x + y = y + x$, $\forall x, y \in V$
2. **Associative:** $(x + y) + z = x + (y + z)$, $\forall x, y, z \in V$
3. **Zero element:** $\exists! 0 \in V$ such that $0 + x = x$, $\forall x \in V$
4. **Inverse:** $\forall x \in V$, $\exists (-x) \in V$ such that $x + (-x) = 0$
5. $(\alpha \beta)x = \alpha(\beta x)$, $\forall \alpha, \beta \in \mathbb{R}$, $x \in V$
6. $\alpha(x + y) = \alpha x + \alpha y$, $\forall \alpha \in \mathbb{R}$, $x, y \in V$
7. $(\alpha + \beta)x = \alpha x + \beta x$, $\forall \alpha, \beta \in \mathbb{R}$, $x \in V$
Examples of Vector Space

1. $\mathbb{R}^n$ with vector sum and scalar multiplication
2. $\mathbb{R}^{m\times n}$: the set of all $m$-by-$n$ matrices
3. $P_n$: the set of all real polynomials in $s$ with degree up to $n$:
   \[
P_n := \{ a_n s^n + \cdots + a_1 s + a_0 \mid a_0, \ldots, a_n \in \mathbb{R} \}
   \]
4. Give an index set $\mathcal{I}$, the set of all mappings from $\mathcal{I}$ to $\mathbb{R}^n$:
   \[
   \mathcal{F}(\mathcal{I}; \mathbb{R}^n) := \{ f : \mathcal{I} \to \mathbb{R}^n \}
   \]
5. $\{ f : \mathbb{R}_+ \to \mathbb{R}^n \mid f$ is differentiable$\}$
6. The set of all functions $f(t)$, $t \geq 0$, with a Laplace transform
7. The set of all square integrable functions $f : \mathbb{R}_+ \to \mathbb{R}$
8. The set of all solutions $x(t) \in \mathbb{R}^n$, $t \geq 0$, to autonomous LTI system
   \[
   \dot{x} = Ax, \quad x(0) = x_0
   \]
Supspaces and Product Spaces

Definition (Subspace)

$W$ is a subspace of vector space $V$ if $W \subseteq V$ and $W$ itself is a vector space under the same vector sum and scalar multiplication operations.

Example:
- $\text{span} \{ v_1, v_2, \ldots, v_k \} := \{ \alpha_1 v_1 + \cdots + \alpha_k v_k : \alpha_i \in \mathbb{R} \} \subset V$
- Diagonal and symmetric matrices
- $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots \subset \mathcal{P}_\infty$

Definition (Product space)

Given two vector spaces $V_1$ and $V_2$, their direct product is the vector space $V_1 \times V_2 := \{ (v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2 \}$

Example:
- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$
- $\mathcal{F}(\mathcal{I}; \mathbb{R}^2) = \mathcal{F}(\mathcal{I}; \mathbb{R}) \times \mathcal{F}(\mathcal{I}; \mathbb{R})$
Bases and Dimension of Vector Spaces

\( v_1, \ldots, v_k \) in vector space \( V \) are linearly independent if for \( \alpha_1, \ldots, \alpha_k \in \mathbb{R} \),

\[
\alpha_1 v_1 + \cdots + \alpha_k v_k = 0 \quad \Rightarrow \quad \alpha_1 = \cdots = \alpha_k = 0
\]

A set of vectors \( \{v_1, \ldots, v_k\} \) is a basis of the vector space \( V \) if
- \( v_1, \ldots, v_k \) are linearly independent in \( V \)
- \( V = \text{span} \{v_1, \ldots, v_k\} \)

Or equivalently,
- each \( v \in V \) has a unique expression \( v = \alpha_1 v_1 + \cdots + \alpha_k v_k \)
- \((\alpha_1, \ldots, \alpha_k)\) is the coordinate of \( v \) in this basis

**Definition (Dimension)**

The dimension of a vector space \( V \) is the number of vectors in any of its basis, and is denoted \( \text{dim} \ V \).

Examples of finite and infinite dimensional vector spaces:
Linear Maps

A map \( f : V \rightarrow W \) between two vector spaces \( V \) and \( W \) is linear if
\[
f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)
\]

Example:

- \( x \in \mathbb{R}^n \mapsto Ax \in \mathbb{R}^m \) for some matrix \( A \in \mathbb{R}^{m \times n} \)
- Projection \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mapsto x_i \in \mathbb{R} \)
- \( X \in \mathbb{R}^{m \times n} \mapsto X^T \in \mathbb{R}^{n \times m} \)
- \( X \in \mathbb{R}^{n \times n} \mapsto A_1 X + X A_2 \in \mathbb{R}^{n \times n} \) for constant \( A_1, A_2 \in \mathbb{R}^{n \times n} \)
- A continuous function on \([0, 1] \mapsto \int_0^t f(x) \, dx \in \mathbb{R} \)
- Polynomial \( p(s) \in \mathcal{P}_n \mapsto p'(s) \in \mathcal{P}_{n-1} \)
- Solutions (zero-state, zero-input responses) of an LTI system

A linear map \( f : V \rightarrow W \) must map \( 0 \in V \) to \( 0 \in W \)

The composition of two linear maps \( f : V \rightarrow W \) and \( g : W \rightarrow U \) is also linear: \( g \circ f : v \in V \mapsto g(f(v)) \in U \)
One-To-One Mapping

Matrix $A \in \mathbb{R}^{m \times n}$ considered as a linear map $\mathbb{R}^n$ to $\mathbb{R}^m$ has null space $\mathcal{N}(A) = \{ x \in \mathbb{R}^n | Ax = 0 \}$

- Set of all vectors orthogonal to all rows of $A$
- Characterize ambiguity in solving equation $Ax = y$

$A \in \mathbb{R}^{m \times n}$ is one-to-one if and only if
- Columns of $A$ are linearly independent
- Rows of $A$ span $\mathbb{R}^n$
- $A$ has rank $n$ (full column rank)
- $A$ has a left inverse: $\exists B \in \mathbb{R}^{n \times m}$ such that $BA = I_n$
Matrix Rank

The rank of a matrix $A \in \mathbb{R}^{m \times n}$ is its maximum number of linearly independent columns (or rows), or equivalently, $\dim \mathcal{R}(A)$

- $\text{Rank} (A) \leq \min(m, n)$
- $\text{Rank} (A) = \text{Rank} (A^T)$
- $\text{Rank} (A) + \dim \mathcal{N}(A) = n$ (conservation of dimension)

Matrix $A \in \mathbb{R}^{m \times n}$ is full rank if $\text{Rank} (A) = \min(m, n)$, which means

- (for skinny matrices) independent column or injective maps
- (for fat matrices) independent rows or surjective maps
- (for square matrices) nonsingular or bijective maps
Matrix Transpose

When $A \in \mathbb{R}^{m \times n}$ is considered as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, its transpose $A^T \in \mathbb{R}^{n \times m}$ is a linear map from $\mathbb{R}^m$ back to $\mathbb{R}^n$.

The following are equivalent:

1. $A$ is one-to-one
2. $A^T$ is onto
3. $\det A^T A \neq 0$
4. $A^T A \in \mathbb{R}^{n \times n}$ is bijective

The following are equivalent:

1. $A$ is onto
2. $A^T$ is one-to-one
3. $\det AA^T \neq 0$
4. $AA^T \in \mathbb{R}^{m \times m}$ is bijective

More generally, for any $A \in \mathbb{R}^{m \times n}$:

- $\mathcal{R}(A^T) = \mathcal{N}(A)^\perp$
- $\mathcal{N}(A^T) = \mathcal{R}(A)^\perp$
Inner Products

For \( x, y \in \mathbb{R}^n \), their inner product is

\[
\langle x, y \rangle := x^T y = x_1 y_1 + \cdots + x_n y_n
\]

For \( x, y, z \in \mathbb{R}^n \)

- \( \langle x, y \rangle = \langle y, x \rangle \)
- \( \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \)
- \( \langle x + y, z \rangle = \langle x, z \rangle + \langle x, y \rangle \)
- \( \langle x, x \rangle = \|x\|^2 \geq 0 \), where \( \|x\| \) is the Euclidean norm of \( x \):

\[
\|x\| := \sqrt{x^T x} = \sqrt{x_1^2 + \cdots + x_n^2}
\]

**Theorem (Cauchy-Schwartz Inequality)**

\[
|\langle x, y \rangle| \leq \|x\| \cdot \|y\|, \quad \forall x, y \in \mathbb{R}^n
\]
Eigenvalues and Eigenvectors

**Eigenvalues/Eigenvectors of a matrix**

- **Evalues/vectors are only defined for square**\(^1\) **matrices**
- For a matrix \( A \in \mathbb{R}^{n \times n} \), we always have \( n \) evalues/evectors
  - Some of these evalues might be distinct, real, repeated, imaginary
  - To find evalues(\( A \)), solve this equation (\( I_n \): identity matrix of size \( n \))
    \[
    \text{det}(\lambda I_n - A) = 0 \quad \text{or} \quad \text{det}(A - \lambda I_n) = 0 \Rightarrow a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0
    \]

- **Example**: \( \text{det} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc. \)

- **Eigenvectors**: A number \( \lambda \) and a non-zero vector \( v \) satisfying
  \[
  Av = \lambda v \Rightarrow (A - \lambda I_n)v = 0
  \]
  are called an eigenvalue and an eigenvector of \( A \)

- \( \lambda \) is an eigenvalue of an \( n \times n \)-matrix \( A \) if and only if \( \lambda I_n - A \) is not invertible, which is equivalent to
  \[
  \text{det}(A - \lambda I_n) = 0.
  \]

---
\(^1\)A square matrix has equal number of rows and columns.
Matrix Inverse

- Inverse of a generic 2by2 matrix:

\[
A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

- Notice that \( A^{-1}A = AA^{-1} = I_2 \)

- Inverse of a generic 3by3 matrix:

\[
A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}^T = \frac{1}{\det(A)} \begin{bmatrix} A & D & G \\ B & E & H \\ C & F & I \end{bmatrix}
\]

\[
A = (ei - fh) \quad D = -(bi - ch) \quad G = (bf - ce) \\
B = -(di - fg) \quad E = (ai - cg) \quad H = -(af - cd) \\
C = (dh - eg) \quad F = -(ah - bg) \quad I = (ae - bd)
\]

\[
\det(A) = aA + bB + cC.
\]

- Notice that \( A^{-1}A = AA^{-1} = I_3 \)
Find the eigenvalues, eigenvectors, and inverse of matrix

\[ A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \]

- Eigenvalues: \( \lambda_{1,2} = 5, -2 \)
- Eigenvectors: \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^\top, \mathbf{v}_2 = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}^\top \)
- Inverse: \( A^{-1} = -\frac{1}{10} \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix} \)

Write \( A \) in the matrix diagonal transformation, i.e., \( A = TDT^{-1} \) where \( D \) is the diagonal matrix containing the eigenvalues of \( A \):

\[
A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}^{-1}
\]

- Only valid for matrices with distinct, real eigenvalues
Rank of a Matrix

- Rank of a matrix: \( \text{rank}(A) \) is equal to the number of linearly independent rows or columns

  - **Example 1:** \( \text{rank}\left( \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 0 & -2 \\ 1 & 2 & 1 \end{bmatrix} \right) = ? \)

  - **Example 2:** \( \text{rank}\left( \begin{bmatrix} -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \right) = ? \)

- Rank computation: reduce the matrix to a simpler form, generally row echelon form, by elementary row operations

  - **Example 2 Solution:**

    \[
    \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \rightarrow 2r_1 + r_2 \rightarrow -3r_1 + r_3 \rightarrow r_2 + r_3 \rightarrow -2r_2 + r_1 \Rightarrow \text{rank}(A) = 2
    \]

- For a matrix \( A \in \mathbb{R}^{m \times n} \): \( \text{rank}(A) \leq \min(m, n) \)

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Null Space of a Matrix

- The Null Space of any matrix $A$ is the subspace $\mathcal{K}$ defined as follows:
  \[ N(A) = \text{Null}(A) = \ker(A) = \{ \mathbf{x} \in \mathcal{K} | A\mathbf{x} = \mathbf{0} \} \]

- Null$(A)$ has the following three properties:
  - Null$(A)$ always contains the zero vector, since $A\mathbf{0} = \mathbf{0}$
  - If $\mathbf{x} \in \text{Null}(A)$ and $\mathbf{y} \in \text{Null}(A)$, then $\mathbf{x} + \mathbf{y} \in \text{Null}(A)$
  - If $\mathbf{x} \in \text{Null}(A)$ and $c$ is a scalar, then $c\mathbf{x} \in \text{Null}(A)$

- **Example:** Find $N(A)$

  \[
  A = \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 5 & 0 \\ -4 & 2 & 3 & 0 \end{bmatrix} \Rightarrow
  \]

  \[
  \begin{bmatrix} 1 & 0 & 1/16 & 0 \\ 0 & 1 & 13/8 & 0 \end{bmatrix} \Rightarrow a = -\frac{1}{16}c, \quad b = -\frac{13}{8}c \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha \begin{bmatrix} -1/16 \\ -13/8 \\ 1 \end{bmatrix} = \tilde{\alpha} \begin{bmatrix} -1 \\ -26 \\ 16 \end{bmatrix}
  \]
Linear Algebra — Example 2

Find the determinant, rank, and null-space set of this matrix:

\[
B = \begin{bmatrix}
0 & 1 & 2 \\
1 & 2 & 1 \\
2 & 7 & 8
\end{bmatrix}
\]

- \(\det(B) = 0\)
- \(\text{rank}(B) = 2\)
- \(\text{null}(B) = \alpha \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \forall \alpha \in \mathbb{R}\)

Is there a relationship between the determinant and the rank of a matrix?
- Yes! Matrix drops rank if determinant = zero \(\Rightarrow\) 1 zero eigenvalue

True or False?
- \(AB = BA\) for all \(A\) and \(B\)—FALSE!
- \(A\) and \(B\) are invertible \(\Rightarrow\) \((A + B)\) is invertible—FALSE!
Matrix Exponential — 1

- Exponential of scalar variable:

\[ e^a = \sum_{i=0}^{\infty} \frac{a^i}{i!} = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \frac{a^4}{4!} + \cdots \]

- Power series converges \( \forall \ a \in \mathbb{R} \)

- How about matrices? For \( A \in \mathbb{R}^{n \times n} \), matrix exponential:

\[ e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!} = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots \]

- What if we have a time-variable?

\[ e^{tA} = \sum_{i=0}^{\infty} \frac{(tA)^i}{i!} = I_n + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + \frac{(tA)^4}{4!} + \cdots \]
Matrix Exponential Properties

For a matrix $A \in \mathbb{R}^{n \times n}$ and a constant $t \in \mathbb{R}$:

1. $A \mathbf{v} = \lambda \mathbf{v} \implies e^{At} \mathbf{v} = e^{\lambda t} \mathbf{v}$
2. $\det(e^{At}) = e^{(\text{trace}(A))t}$
3. $(e^{At})^{-1} = e^{-At}$
4. $e^{A^\top t} = (e^{At})^\top$
5. If $A$, $B$ commute, then: $e^{(A+B)t} = e^{At} e^{Bt} = e^{Bt} e^{At}$
6. $e^{A(t_1+t_2)} = e^{At_1} e^{At_2} = e^{At_2} e^{At_1}$

---

Trace of a matrix is the sum of its diagonal entries.
When Is It Easy to Find $e^A$? Method 1

Well...Obviously if we can directly use $e^A = I_n + A + \frac{A^2}{2!} + \cdots$

**Three cases for Method 1**

**Case 1** $A$ is nilpotent$^3$, i.e., $A^k = 0$ for some $k$. Example:

\[
A = \begin{bmatrix}
5 & -3 & 2 \\
15 & -9 & 6 \\
10 & -6 & 4
\end{bmatrix}
\]

**Case 2** $A$ is idempotent, i.e., $A^2 = A$. Example:

\[
A = \begin{bmatrix}
2 & -2 & -4 \\
-1 & 3 & 4 \\
1 & -2 & -3
\end{bmatrix}
\]

**Case 3** $A$ is of rank one: $A = uv^T$ for $u, v \in \mathbb{R}^n$

\[
A^k = (v^Tu)^{k-1}A, \ k = 1, 2, \ldots
\]

$^3$Any triangular matrix with 0s along the main diagonal is nilpotent
Method 2 — Jordan Canonical Form

All matrices, whether diagonalizable or not, have a Jordan canonical form: $A = T J T^{-1}$, then $e^{At} = T e^{Jt} T^{-1}$

Generally, $J = \begin{bmatrix} J_1 & \cdots & \cdots \\ \vdots & \ddots & \ddots \\ \vdots & \cdots & \ddots \end{bmatrix}$

Thus, $J_i = \begin{bmatrix} \lambda_i & 1 & \cdots \\ \vdots & \ddots & \ddots \\ \vdots & \cdots & \ddots \end{bmatrix} \in \mathbb{R}^{n_i \times n_i}$

$e^{J_{it}} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{n_i - 1}e^{\lambda_i t}}{(n_i - 1)!} \\ 0 & e^{\lambda_i t} & \cdots & \frac{t^{n_i - 2}e^{\lambda_i t}}{(n_i - 2)!} \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\lambda_i t} \end{bmatrix} \Rightarrow e^{At} = T \begin{bmatrix} e^{J_{1t}} & \cdots & \cdots \\ \vdots & \ddots & \ddots \\ \vdots & \cdots & e^{J_{ot}} \end{bmatrix} T^{-1}$

Jordan blocks and marginal stability
Examples

- Find $e^{A(t-t_0)}$ for matrix $A$ given by:

$$A = TJT^{-1} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^{-1}$$

- Solution:

$$e^{A(t-t_0)} = T e^{J(t-t_0)} T^{-1}$$

$$= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} e^{-(t-t_0)} & 0 & 0 & 0 \\ 0 & 1 & t-t_0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-(t-t_0)} \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^{-1}$$

- Find $e^{A(t-t_0)}$ for matrix $A$ given by:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$
Jordan Canonical Form

**Theorem (Jordan Canonical Form)**

For any $A \in \mathbb{R}^{n \times n}$, there exists a nonsingular $T \in \mathbb{C}^{n \times n}$ such that

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix}, \quad J_i = \begin{bmatrix} \lambda_i & 1 \\ & \ddots & \ddots \\ & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i}$$

- Unique up to permutation of Jordan blocks
- Diagonalizable matrices are special cases with all $n_i = 1$

**Definition (Algebraic and Geometric Multiplicity)**

The algebraic multiplicity of an eigenvalue $\lambda_i$ is the sum of the sizes of all Jordan blocks corresponding to it; its geometric multiplicity is the number of all such Jordan blocks.
Finding Jordan Canonical Form

1. The objective here is to show how to find $A = T J T^{-1}$ for a nondiagonalizable matrix $A$

2. Assume that matrix $A$ has $n$ eigenvalues
   - $k$ values are distinct AND not repeated (multiplicity = 1, $\lambda_1, \lambda_2, \ldots, \lambda_k$)
   - Hence, there are $n - k$ values that are repeated (multiplicity $\geq 2$)

3. First, Find the $k$ eigenvectors relating to these eigenvalues and list the first $k$ eigenvalues on the first $k$ diagonal entries of $J$. Also, group the first $k$ eigenvectors in the first $k$ columns of $T$

4. What’s left now: $n - k$ generalized eigenvectors of the other values that are repeated at least twice, and the Jordan blocks corresponding to these values

5. Assume that out of the $n - k$ values, there are $m$ distinct ones

6. Find the eigenvectors that correspond to the $m$ distinct ones—you should obtain at least $m$ eigenvectors

7. What’s left now: find the other generalized eigenvectors (i.e., $n - k - m$ eigenvectors) and Jordan blocks (number of Jordan blocks corresponding to the repeated values is equal to the number of linearly independent eigenvectors)
Example: find the Jordan canonical form of this matrix

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 \\
1 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\pi_A(\lambda) = \lambda^4(\lambda - 1) = 0
\]

Two eigenvalues: \(\lambda_1 = 1\) (not repeated), \(\lambda_2 = 0\) (repeated 4 times)

First: find evector for \(\lambda_1 = 1\)

\[
(A - \lambda_1 I_5)\mathbf{v}_1 = 0 \Rightarrow \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & -1 \\
1 & -1 & -1 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 \\
-1 & 1 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\mathbf{v}_1 \\
\end{bmatrix} = 0 \Rightarrow \mathbf{v}_1 = [1, 1, 1, 1, -1]^T
\]

Now, let’s find the generalized evec tors for \(\lambda_2 = 0\) and the associated Jordan block. Note that the \(A\) matrix is of rank 3

First, find the LI evec tors of \(\lambda_2\):

\[
(A - \lambda_2 I_5)\mathbf{v}_2 = 0 \Rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 \\
1 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & -1 \\
-1 & 1 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
\mathbf{v}_2 \\
\end{bmatrix} = 0 \Rightarrow \mathbf{v}_2 \in \mathcal{N}(A)
\]
You can see that \( \nu_2 \) actually spans two column vectors since \( A \) is of rank 3.

The two LI eigenvectors generated from \( A\nu_2 = 0 \) are:

\[
\nu_2^1 = [0 \ 0 \ 0 \ 1 \ 0]^\top, \nu_2^2 = [0 \ 0 \ -1 \ 0 \ 0]^\top
\]

Therefore, we have two Jordan blocks corresponding to \( \lambda_2 \).

We have two alternatives for the sizes these two Jordan blocks: either (3,1) or (2,2).

How do we know the correct size?

The number of Jordan blocks of size exactly \( j \) is

\[
2 \dim \ker(A - \lambda_i I)^j - \dim \ker(A - \lambda_i I)^{j+1} - \dim \ker(A - \lambda_i I)^{j-1}
\]

Hence, the number of Jordan blocks of size 1 is: \( 2 \times 2 - 3 - 0 = 1 \), hence the size the Jordan blocks of size 3 is also one, which means (3,1) is a legit Jordan block sizes.

\[ \Rightarrow J = ? \]

Now that we have the Jordan blocks, we need to find the two other generalized eigenvectors corresponding to \( \nu_2^2 \).
Examples

- Find \( e^{A(t-t_0)} \) for matrix \( A \) given by:

\[
A = T J T^{-1} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^{-1}
\]

Solution:

\[
e^{A(t-t_0)} = T e^{J(t-t_0)} T^{-1}
\]

\[
= \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix}
e^{-(t-t_0)} & 0 & 0 & 0 \\
0 & 1 & t - t_0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{-(t-t_0)}
\end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}^{-1}
\]

- Find \( e^{A(t-t_0)} \) for matrix \( A \) given by:

\[
A_1 = \begin{bmatrix} 1 & 0 \\
0 & -2 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\
0 & -2 \end{bmatrix}
\]
Solution to the State-Space Equation

- In the next few slides, we’ll answer this question: what is a solution to this vector-matrix first order ODE:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{align*}
\]

- By solution, we mean a closed-form solution for \(x(t)\) and \(y(t)\) given:
  - An initial condition for the system, i.e., \(x(t_{\text{initial}}) = x(0)\)
  - A given control input signal, \(u(t)\), such as a step-input \((u(t) = 1)\), ramp \((u(t) = t)\), or anything else
The Curious Case of Autonomous Systems—Case 1

Let’s assume that we seek solution to this system first:

\[
\dot{x}(t) = Ax(t), \quad x(0) = x_0 = \text{given} \\
y(t) = Cx(t)
\]

This means that the system operates without any control input—autonomous system (e.g., autonomous vehicles)

First, let’s look at \( \dot{x}(t) = Ax(t) \)—what’s the solution to this first order ODE?

- First case: \( A = a \) is a scalar \( \Rightarrow x(t) = e^{at}x_0 \)
- Second case: \( A \) is a matrix

\[
\Rightarrow x(t) = e^{At}x_0 \Rightarrow y(t) = Cx(t) = Ce^{At}x_0
\]

Exponential of scalars is very easy, but exponentials of matrices can be very challenging

Hence, for an \( n \)th order system, where \( n \geq 2 \), we need to compute the matrix exponential in order to get a solution for the above system—we learned that in the linear algebra revision section
Example (Case 1)

\[ x(t) = e^{At}x_0, \quad y(t) =Cx(t) = Ce^{At}x_0 \]

- Find the solution for these two autonomous systems separately:

  \[ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad x_0^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

  \[ A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & 0 \end{bmatrix}, \quad x_0^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

- Note that this system is diagonalizable (Case A)

- If the system is not diagonalizable, we have to look for other methods to find the matrix exponential

- In particular, we have to find the Jordan form

- Anyway, let’s find the state and output solutions now for this diagonalizable system

**Solution:**
Case 2—Systems with Inputs

- MIMO (or SISO) LTI dynamical system:
  \[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_{t_0} = \text{given} \]
  \[ y(t) = Cx(t) + Du(t) \]

- The solution to the above ODE is given by:
  \[ x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau) \, d\tau \]

- Clearly the output solution is:
  \[ y(t) = C \left( e^{A(t-t_0)}x_{t_0} \right) + C \left( \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau) \, d\tau \right) + Du(t) \]

- **Question**: how do I analytically compute \( y(t) \) and \( x(t) \)?

- **Answer**: you need to (a) integrate and (b) compute matrix exponentials (given \( A, B, C, D, x_{t_0}, u(t) \))
Example (Case 2)

\[
x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau) \, d\tau
\]

\[
y(t) = C \left( e^{A(t-t_0)}x_{t_0} \right) + C \left( \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau) \, d\tau \right) + Du(t)
\]

- zero input response
- zero state response

Find the solution for these two LTI systems with inputs:

\[
A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad x_0^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad D_1 = 0, \quad u_1(t) = 1
\]

\[
A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 & 0 \end{bmatrix}, \quad x_0^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad D_2 = 1, \quad u_2(t) = 2e^{-2t}
\]

Solution:
Questions And Suggestions?

Any questions?

Thank You!

Please visit engineering.utsa.edu/~taha

IFF you want to know more 😊