Module 02
CPS Background: Linear Systems Preliminaries

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EE 5243: Introduction to Cyber-Physical Systems

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Objective: assess students’ knowledge of the course prerequisites

You are not supposed to do so much work for this exam — it’s only assessment, remember!

You will receive a perfect grade whether you know the answers or not
Find the eigenvalues, eigenvectors, and inverse of matrix

\[ A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \]

**Answers**
- Eigenvalues: \( \lambda_{1,2} = 5, -2 \)
- Eigenvectors: \( v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^\top, v_2 = \begin{bmatrix} -\frac{4}{3} \\ 1 \end{bmatrix}^\top \)
- Inverse: \( A^{-1} = -\frac{1}{10} \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix} \)

Write \( A \) in the matrix diagonal transformation, i.e., \( A = TDT^{-1} \) — \( D \) is the diagonal matrix containing the eigenvalues of \( A \).

**Answers**
- Diagonal Transformation: \( A = \begin{bmatrix} v_1 & v_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}^{-1} \)
Find the determinant, rank, and null-space set of this matrix:

\[
B = \begin{bmatrix}
0 & 1 & 2 \\
1 & 2 & 1 \\
2 & 7 & 8
\end{bmatrix}
\]

- **Answers**
  - \( \det(B) = 0 \)
  - \( \text{rank}(B) = 2 \)
  - \( \text{null}(B) = \alpha \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \forall \alpha \in \mathbb{R} \)

Is there a relationship between the determinant and the rank of a matrix?

- **Answer**
  - Yes! Matrix drops rank if determinant = zero \( \rightarrow \) minimal of 1 zero value

**True or False?**

- \( AB = BA \) for all \( A \) and \( B \)
- \( A \) and \( B \) are invertible \( \rightarrow (A + B) \) is invertible
LTI Systems — 1

- LTI dynamical system can be represented as follows:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x_{\text{initial}} = x_{t_0}, \quad (1) \\
y(t) &= Cx(t) + Du(t), \quad (2)
\end{align*}
\]

- \(x(t)\): dynamic state-vector of the LTI system, \(u(t)\): control input-vector
- \(y(t)\): output-vector and \(A, B, C, D\) are constant matrices

What is the closed-form, state solution to the above differential equation for any time-varying control input, i.e., \(x(t) =?\)

**Answer:**

\[
x(t) = e^{A(t-t_0)}x_{t_0} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau) \, d\tau
\]

*Can you verify that the above solution is actually correct?* Hint:

\[
\frac{d}{d\theta} \left( \int_{a(\theta)}^{b(\theta)} f(x, \theta) \, dx \right) = \int_{a(\theta)}^{b(\theta)} \partial_x f(x, \theta) \, dx + f(b(\theta), \theta) \cdot b'(\theta) - f(a(\theta), \theta) \cdot a'(\theta)
\]

Leibniz Differentiation Theorem
What are the poles of the above dynamical system? Define asymptotic stability. What’s the difference between marginal and asymptotic stability?

**Answers:**
- Poles: \( \text{eig}(A) \)
- Asymptotic stability: poles \textbf{strictly} in the LHP
- Marginal stability: some poles can be on the imaginary axis

What is the transfer function \( H(s) = \frac{Y(s)}{U(s)} \) of the above system?

**Answers:**
- TF:
  \[
  H(s) = C(sI - A)^{-1}B + D
  \]
  Above TF valid for MIMO systems — it becomes a TF matrix, rather than a scalar quantity

See Chen [1995] for more details
What is the state-transition matrix for the above system?

**Answer:**
- $\phi(t, t_0) = e^{A(t-t_0)}$
- How about linear, time-varying systems?

Under what conditions is the above dynamical system controllable? Observable? Stabilizable? Detectable?

**Answers:**
- Controllability
- Observability
- Stabilizability
- Detectability
Define the linear quadratic regulator problem for LTI systems in both, words and equations

**Answers:**
- Objective: minimize the total cost of state-deviation and consumed control (i.e., taking control actions)
- Constraints: state-dynamics, control inputs, initial conditions
- Equations:

\[
\begin{align*}
\text{minimize} & \quad J = x^T(t_1)F(t_1)x(t_1) + \int_{t_0}^{t_1} (x^TQx + u^TRu + 2x^TNu) \, dt \\
\text{subject to} & \quad \dot{x}(t) = Ax(t) + Bu(t) \\
& \quad x(t_i) = x_{t_i} \quad (3) \\
& \quad u \in U, \ x \in X 
\end{align*}
\]
Optimal Control and Dynamic Observers — 2

- What is the optimal solution to an optimal control problem? What does it physically mean for CPSs?

**Answers:**
- It’s the optimal trajectory of the control input and the corresponding state-trajectory
- Physical meaning: *you’re better off selecting this control input, among all other — possibly infinite — control input alternatives*

- What is a generic dynamic observer? Luenberger observer?

**Answer:**
- An estimator for internal states of the system
Optimization — 1

- Let \( f(x) \) be a multi-variable function of three variables, as follows:

\[
f(x_1, x_2, x_3) = x_1 x_2 x_3 + 2x_2^3 x_1^2 - 4 \cos(x_3 x_2) + \log(\cos(x_2)^2) + 4x_3 - 2x_1
\]

- Find the Jacobian and Hessian of \( f(x) \)

- **Answers:**

\[
\nabla f = \begin{bmatrix}
4x_1 x_2^3 + x_3 x_2 - 2 \\
6x_1^2 x_2^2 + x_1 x_3 - (2 \sin(x_2))/\cos(x_2) + 4x_3 \sin(x_2 x_3)
\end{bmatrix}, \\
x_1 x_2 + 4x_2 \sin(x_2 x_3) + 4
\]

\[
\nabla^2 f = \begin{bmatrix}
4x_2^3 & 12x_1 x_2^2 + x_3 \\
12x_1 x_2 + 3x_2 & 12x_2^2 x_2 - (2 \sin(x_2)^2)/\cos(x_2)^2 + 4x_3^2 \cos(x_2 x_3) - 2
\end{bmatrix}
\]

\[
x_1 + 4 \sin(x_2 x_3) + 4x_2 x_3 \cos(x_2 x_3) \\
x_1 + 4 \sin(x_2 x_3) + 4x_2 x_3 \cos(x_2 x_3)
\]
Optimization — 2

- What is an optimization problem? An unfeasible, feasible solution, an optimal solution to a generic optimization problem?

- What is a convex optimization problem? Define it rigorously

**Answers:**

- An optimization problem of finding some $x^* \in \mathcal{X}$ such that:
  \[
  f(x^*) = \min \{ f(x) : x \in \mathcal{X} \}
  \]

  * $\mathcal{X} \subset \mathbb{R}^n$ is the feasible set and $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective, is called convex if $\mathcal{X}$ is a closed convex set and $f(x)$ is convex on $\mathbb{R}^n$

- Alternatively, convex optimization problems can be written as:
  \[
  \begin{align*}
  \text{minimize} & \quad f(x) \\
  \text{subject to} & \quad g_i(x) \leq 0, \quad i = 1, \ldots, m
  \end{align*}
  \]

  * $f, g_1, \ldots, g_m : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex

- See [Boyd & Vandenberghe, 2004] for more
What is a semi-definite program (SDP)? Define it rigorously.

**Answers:**
- A semidefinite program minimizes a linear cost function of the optimization variable \( z \in \mathbb{R}^n \) subject to a matrix inequality condition.
- An SDP can formulated as follows:

\[
\begin{align*}
\text{minimize} & \quad f(z) \\
\text{subject to} & \quad F(z) \succeq 0,
\end{align*}
\]

* Where

\[
F(z) \triangleq F_0 + \sum_{i=1}^{n} z_i F_i
\]

Given the following optimization problem,

\[
\begin{align*}
\text{minimize} & \quad c(x) \\
\text{subject to} & \quad h(x) \leq 0 \\
& \quad g(x) = 0,
\end{align*}
\]

what are the corresponding Karush–Kuhn–Tucker (KKT) conditions?
Questions And Suggestions?

Any questions?

Thank You!

Please visit
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IFF you want to know more 😊
References I
