Module 06
Stability of Dynamical Systems

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The following CT LTI system without inputs

\[ \dot{x}(t) = A(t)x(t), \quad x(t) \in \mathbb{R}^n \]

has an equilibrium at \( x_e = 0 \).

**Asymptotic Stability**

The above system is asymptotically stable at \( x_e = 0 \) if its solution \( x(t) \) starting from any initial condition \( x(t_0) \) satisfies

\[ x(t) \to 0, \text{ as } t \to \infty \]

**Exponential Stability**

The above system is exponentially stable at \( x_e = 0 \) if its solution \( x(t) \) starting from any initial condition \( x(t_0) \) satisfies

\[ \|x(t)\| \leq Ke^{-rt}\|x(t_0)\|, \quad \forall t \geq t_0 \]

for some positive constants \( K \) and \( r \).
Example 1

Consider the following TV LTI system from Homework 4:

\[
\dot{x}(t) = \begin{bmatrix}
-\frac{1}{t+1} & 0 \\
-t/t+1 & 0
\end{bmatrix} x(t)
\]

Recall that the solution to this system is

\[
x(t) = \phi(t, 0)x(0) = \begin{bmatrix}
\frac{1}{t+1} & 0 \\
-t/t+1 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
-1/5
\end{bmatrix} = \begin{bmatrix}
\frac{1}{t+1} \\
-t/t+1 - 1/5
\end{bmatrix}
\]

Is this system asymptotically stable?

Solution: it’s not, since the states do not go to zero for any initial conditions
Stability of CT LTI Systems

- For this CT LTI system

\[ \dot{x}(t) = Ax(t) \]

the solution \( x(t) = e^{At}x(t_0) \) is a linear combination of the modes of the system.

- In other words, \( x(t) \) is a linear combinations of \( p(t)e^{\lambda_it} \)

- \( p(t) \) is a polynomial of \( t \)

- Why does that make sense? Well...

Stability of LTI Systems

The following theorems are equivalent:

- The LTI system is asymptotically stable
- The LTI system is exponentially stable
- All eigenvalues of \( A \) are in the open left half of the complex plane
Marginal Stability

Definition of Marginal Stability

The CT LTI system \( \dot{x}(t) = Ax(t) \) is called marginally stable if both of these statements are true:

- All eigenvalues of \( A \) are in the closed\(^a\) LHP
- There are some eigenvalues of \( A \) on the \( j\omega \)-axis, and all the Jordan blocks associated with such eigenvalues have size one

\(^a\)A closed set can be defined as a set which contains all its limit points.

For marginally stable systems:

- Starting from any initial conditions, the solution \( x(t) \) will neither converge to zero nor diverge to infinity
- State solutions will converge (not necessarily to zero) only if all \( j\omega \) \( \) axis are zero
- Can you justify these findings?
- From now on: stability means asymptotic stability
Example 2

Consider the CT LTI system with \( A = \begin{bmatrix} -12 & -4 \\ -2 & -1 \end{bmatrix} \)

This system has eigenvalues \( \lambda_1 = -12.68, \lambda_2 = -0.31 \)

The two eigenvalues are in the LHP

Hence, the system is asymptotically stable
Unstable LTI Systems

Definition of Instability

The CT LTI system $\dot{x}(t) = Ax(t)$ is unstable if either of these statements is true:

- $A$ has an eigenvalue (or eigenvalues) in the open RHP
- $A$ has an eigenvalues on the $j\omega$-axis whose at least one Jordan block has size greater than one

*This means that the state solutions will diverge to infinity
Example 3

\[
\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)
\]

- Is this system stable?
- From Homework 3, the state solution is (we solved for initial conditions \( x(1) \)):

\[
x(t) = e^{A(t-1)}x(1) = \begin{bmatrix} 1 \\ 2t - 1 \\ 6t^2 - 6t + 1 \end{bmatrix}
\]

- Clearly, this system is unstable
- Eigenvalues are all equal to zero, and the size of Jordan blocks is three (greater than 1)
Stability of LTV Systems

- We talked about asymptotic and exponential stability
- These concepts are easy to verify for LTI systems
- What about CT LTV systems? What are the eigenvalues of \( A(t) \)?
- You cannot often answer this question
- Solution? **Find the STM**
- Recall that \( x(t) = \phi(t, t_0)x(t_0) \) for LTV systems
- **System is asymptotically stable iff** \( \phi(t, t_0) \to 0 \) as \( t \to \infty \)
- **System is exponentially stable iff** there exist positive constants \( C, r \) such that
  \[
  \|\phi(t, t_0)\| \leq Ce^{-rt}
  \]
  for all \( t \geq t_0 \)
- For LTV systems, asymptotic stability is **not** equivalent to exponential stability
Example 2

Consider the following TV LTI system from Homework 4:

\[
\dot{x}(t) = \begin{bmatrix}
-1 + \cos(t) & 0 \\
-2 + \sin(t) & 0
\end{bmatrix} x(t)
\]

The state transition matrix is:

\[
\phi(t, t_0) = \begin{bmatrix}
e^{-2(t-t_0)+\cos(t_0)-\sin(t)} & 0 \\
0 & e^{-2(t-t_0)+\cos(t_0)-\sin(t)}
\end{bmatrix}
\]

Is this system exponentially stable?

**Solution:** We'll have to prove that

\[
\|x(t)\| \leq K e^{-r t} \|x(t_0)\|, \quad \forall t \geq t_0
\]

and basically obtain \(K\) and \(r\)

Note that:

\[
\|\phi(t, t_0) x(t_0)\| \leq \|\phi(t, t_0)\| \|x(t_0)\|
\]

and

\[
|e^{-(t-t_0)+\sin(t)-\sin(t_0)}| = |e^{-(t-t_0)}| \cdot |e^{\sin(t)-\sin(t_0)}| \leq e^2 e^{-(t-t_0)}
\]

\[
|e^{-2(t-t_0)+\cos(t_0)-\cos(t)}| = |e^{-2(t-t_0)}| \cdot |e^{\cos(t_0)-\cos(t)}| \leq e^2 e^{-2(t-t_0)}
\]

Hence, we can extract \(K\) and \(r\) given the norm of \(\phi(t, t_0)\):

\[
K = e^2 \cdot e^{t_0}, \quad r = 1
\]
Stability of DT LTV Systems

Consider the following DT LTI system

\[ x(k + 1) = A(k)x(k), \quad x(k) \in \mathbb{R}^n \]

**Asymptotic Stability**

The above system is asymptotically stable at time \( k_0 \) its solution \( x[k] \) starting from any initial condition \( x(k_0) \) at time \( k_0 \) satisfies:

\[ x(k) \to 0, \text{ as } k \to \infty \]

**Exponential Stability**

The above system is exponentially stable at time \( k_0 \) its solution \( x[k] \) starting from any initial condition \( x(k_0) \) at time \( k_0 \) satisfies:

\[ \|x(k)\| \leq Kr^{k-k_0}\|x(k_0)\|, \quad \forall k = k_0, k_0 + 1, k_0 + 2, \ldots \]

for some constants \( K > 0 \) and \( 0 \leq r < 1 \).
Stability of DT LTI Systems

For this DT LTI system

\[ x(k + 1) = Ax(k) \]

the following theorems are equivalent:

- The DT LTI system is asymptotically stable
- The DT LTI system is exponentially stable
- All eigenvalues of \( A \) are inside the open unit disk of the complex plane
Example 4

\[ x(k + 1) = \begin{bmatrix} 0.5 & 0.3 \\ 0 & -0.4 \end{bmatrix} x(k) \]

- This system has two eigenvalues:
  \[ \lambda_1 = 0.5, \lambda_2 = -0.4 \]
- Both are inside the unit disk, hence the system is stable
Marginal Stability, Instability of DT LTI Systems

Definition of Marginal Stability

The DT LTI system $x(k+1) = Ax(k)$ is called marginally stable if both of these statements are true:

- All eigenvalues of $A$ are inside the closed unit disk.
- There are some eigenvalues of $A$ on the unit circle, and all the Jordan blocks associated with such eigenvalues have size one.

Definition of Instability

The DT LTI system $x(k+1) = Ax(k)$ is unstable if either of these statements is true:

- $A$ has an eigenvalue (or eigenvalues) outside the closed unit disk.
- $A$ has an eigenvalues on the unit circle whose at least one Jordan block has size greater than one.
Example 5

\[ x(k + 1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(k), x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \]

- This system has two eigenvalues: \( \lambda_1 = j, \lambda_2 = -j \)
- These eigenvalues are located on the boundaries of the unit disk
- The state solution is given:
  \[ x_1(k) = x_{10} \cos(0.5k\pi) + x_{20} \sin(0.5k\pi) \]
  \[ x_2(k) = x_{20} \cos(0.5k\pi) + x_{10} \sin(0.5k\pi) \]
- For any \( x(0) \), this system will be marginally stable
Stability of DT LTV Systems

- For DT LTV systems, asymptotic stability is **not** equivalent to exponential stability.
- Recall that $x(k) = \phi(k, k_0)x(k_0)$ for DT LTV systems $(x(k + 1) = A(k)x(k))$.
- **DT LTV system is asymptotically stable iff** $\phi(k, k_0) \to 0$ as $k \to \infty$.
- **DT LTV system is exponentially stable iff** there exist $C > 0$ and $0 \leq r < 1$ such that
  \[ \|\phi(k, k_0)\| \leq Cr^{k-k_0} \]
  for all $k \geq k_0$. 

Summary

<table>
<thead>
<tr>
<th>system</th>
<th>continuous-time</th>
<th>discrete-time</th>
</tr>
</thead>
<tbody>
<tr>
<td>asympt. stable</td>
<td>$\forall i = 1, \ldots, n$ (\Re(\lambda_i) &lt; 0)</td>
<td>$</td>
</tr>
<tr>
<td>unstable</td>
<td>$\exists i$ such that $\Re(\lambda_i) &gt; 0$</td>
<td>$</td>
</tr>
<tr>
<td>stable</td>
<td>$\forall i, \ldots, n$ $\Re(\lambda_i) \leq 0$</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td>and $\forall \lambda_i$ such that algebraic $=$ geometric mult.</td>
<td>$\Re(\lambda_i) = 0$</td>
</tr>
</tbody>
</table>

- In the above table, “stable” means **marginally stable**
Aleksandr Mikhailovich Lyapunov (1857—1918)
Lyapunov methods: very general methods to prove (or disprove) stability of nonlinear systems

Lyapunov’s stability theory is the single most powerful method in stability analysis of nonlinear systems.

Consider a nonlinear system: \( \dot{x}(t) = f(x) \)

- A point \( x_{eq} \) is an equilibrium point if \( f(x_{eq}) = 0 \)
- Can always consider that \( x_0 = 0 \); if not, you can shift coordinates

Any equilibrium point is:

- **Stable in the sense of Lyapunov**: if (arbitrarily) small deviations from the equilibrium result in trajectories that stay (arbitrarily) close to the equilibrium for all \( t \)
- **Asymptotically stable**: if small deviations from the equilibrium are eventually forgotten and the system returns asymptotically to the equilibrium point
- **Exponentially stable**: if the system is asymptotically stable, and the convergence to the equilibrium point is fast
The Math

The equilibrium point is

- **Stable in the sense of Lyapunov (ISL)** (or simply stable) if for each \( \epsilon \geq 0 \), there is \( \delta = \delta(\epsilon) > 0 \) such that
  \[
  ||x(0)|| < \delta \Rightarrow ||x(t)|| \leq \epsilon, \quad \forall t \geq 0
  \]

- **Asymptotically stable** if there exists \( \delta > 0 \) such that
  \[
  ||x(0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0
  \]

- **Exponentially stable** if there exist \( \{\delta, \alpha, \beta\} > 0 \) such that
  \[
  ||x(0)|| < \delta \Rightarrow ||x(t)|| \leq \beta e^{-\alpha t}, \quad \forall t \geq 0
  \]

- **Unstable** if not stable
Stability of nonlinear systems

- ISL or Marginally Stable
- Unstable
More on Lyap Stability

- How do we analyze the stability of an equilibrium point \textbf{locally}?
- Well, for nonlinear systems we can find all equilibrium points (previous modules)
- We can obtain the linearized dynamics $\dot{x}(t) = A_{eq}^{(i)} \dot{x}(t)$ for all equilibria $i = 1, 2, \ldots$
- You can then find the eigenvalues of $A_{eq}^{(i)}$: if all are negative, then that particular equilibrium point is \textbf{locally stable}
- This method is called \textbf{Lyapunov’s first method}
- How about global conclusion for $\dot{x}(t) = f(x(t))$?
- You’ll have to study \textbf{Lyapunov Function} that give you insights on the global stability properties of nonlinear systems
- We can’t cover these in this class
Simple Example

- Analyze the stability of this system

\[ \dot{x}(t) = \frac{2}{1 + x(t)} - x(t) \]

- This system has two equilibrium points:

\[ x_{eq}^{(1)} = 1, \quad x_{eq}^{(2)} = -2 \]

- Analyze stability of each point

- Example 2: the inverted pendulum in the previous lecture
Questions And Suggestions?

Any questions?

Thank You!

Please visit
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IFF you want to know more 😊