Module 06
Higher Order Systems, Stability Analysis &
Steady-State Errors

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EE 3413: Analysis and Design of Control Systems

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Module 6 Outline

1. From FOSs & SOSs to higher-order systems
2. Stability of linear systems
3. Routh-Hurwitz stability criterion
4. System types & steady-state tracking errors
5. Reading sections: 5.4, 5.6, 5.8 Ogata, 5.6, 6.1, 6.2 Dorf and Bishop
Nonstandard SOSs

\[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

- So far, we analyzed the above TFs for SOSs
- What if we have a non-unit DC gain?
  \[ H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]
  - What’s \( y_{\text{step}}(\infty) \)? Behavior won’t change as much
- What if we have a zero:
  \[ H(s) = \frac{\alpha s \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]
  - Given an extra zero, we obtain:
    \[ H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{\alpha s}{s^2 + 2\zeta \omega_n s + \omega_n^2} = H_1(s) + H_2(s) = H_1(s) + \frac{\alpha}{\omega_n^2} s H_1(s) \]
Adding an Extra Zero

\[ H(s) = H_1(s) + H_2(s) = H_1(s) + \frac{\alpha}{\omega_n^2} sH_1(s) \]

- Therefore, under any input (step, impulse, ramp), the response will be:

\[ y(t) = y_1(t) + y_2(t) = y_1(t) + \frac{\alpha}{\omega_n^2} y'_1(t) \]

- \( y_1(t) \): unit-step response of standard SOS; Step response example
- Zero affects overshoot in the step response

\[ H(s) = \frac{s+1}{s^2+0.8s+1} \]
Higher Order Systems

- How can we analyze systems with more zeros, more poles?
- First, write the TF in this standard form:
  \[ H(s) = K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \]
- Location of poles determines almost everything
- How many cases do we have?

(1) For distinct real poles:

\[ H(s) = \frac{\alpha_1}{s - p_1} + \cdots + \frac{\alpha_n}{s - p_n} \]

- Unit step and impulse responses? Easy to derive

\[ y_{imp}(t) = \alpha_1 e^{p_1 t} + \cdots + \alpha_n e^{p_n t}, \quad y_{step}(t) = \beta_0 + \beta_1 e^{p_1 t} + \cdots + \beta_n e^{p_n t} \]

- Transients will vanish if\[ p_1, \ldots, p_n \] are negative

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Mean, Complex Poles

(2) For distinct real and complex poles:

\[ H(s) = \sum_{j=1}^{q} \frac{\alpha_j}{s - p_j} + \sum_{k=1}^{r} \frac{\beta_k s + \gamma_k}{s^2 + 2\sigma_k s + \omega_k^2} \]

- You’ll have to show me your PFR superpowers to obtain \( \alpha_j, \beta_k, \gamma_k, \sigma_k, \omega_k \) \( \forall j, k \)
- Unit-impulse response:

\[ y_{imp}(t) = \sum_{j=1}^{q} \alpha_j e^{p_j t} + \sum_{k=1}^{r} c_k e^{-\sigma_k t} \sin(\omega_k t + \theta_k) \]

- Unit-step response:

\[ y_{step}(t) = \sum_{j=1}^{q} d_j e^{p_j t} + \sum_{k=1}^{r} f_k e^{-\sigma_k t} \sin(\omega_k t + \phi_k) \]

- Similar to the previous case, transients will vanish if all poles are in the LHP
Summary & Important Remarks

- Each real pole $p$ contributes to an exponential term in any response.
- Each complex pair of poles contributes a modulated oscillation.
  - The decay of these oscillations depend on whether the real-part of the pole is negative or positive.
  - The magnitude of oscillations, contributions depends on residues, hence on zeros.
- **Dominant poles**: poles that dominate any kind of output response.
  - Dominant poles can be real (be real ok?) or complex.
Dominant Poles — Example

\[ H_1(s) = \frac{1}{(s^2 + 2s + 2)(s^2 + 8s + 25)} \]

\[ p_{1,2} = -1 \pm j \quad p_{3,4} = -4 \pm j3 \]

\[ H_2(s) = \frac{1/25}{s^2 + 2s + 2} \]

\[ p_{1,2} = -1 \pm j \]
Who Likes Stability? Who Likes Instability?

- Stability: one of the most important problems in control
- System is stable if, under bounded input, its output will converge to a finite value, i.e., transient terms will eventually vanish. Otherwise, it is unstable
- Above definition is a tricky one—we need a quantitative one
- From now on, this system is stable iff all $p$’s have strictly negative real parts

$$H(s) = K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}$$

- If $p_i = 0$, would the system be stable? NO, NO.
Design Problems Related to Stability

- **Stability Criterion**: for a given system (i.e., given $C(s)$, $G(s)$), determine if it is stable.

- **Stabilization**: for a given system that is unstable (i.e., poles of $G(s)$ are unstable), design $C(s)$ such as $\frac{Y(s)}{U(s)}$ is stable.

- Most engineering design applications for control systems evolve around this simple, yet occasionally challenging idea.

- Some systems **cannot be stabilized**.

- For more complex $G(s)$, design of $C(s)$ is likely to be more complex.

- However, this **IS NOT A RULE**.
How to Infer Stability? Two Methods

\[ H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n} \]

- System, denoted by the above TF \( H(s) \) is stable iff:
  \[ \text{roots}(a_0 s^n + a_1 s^{n-1} + \cdots + a_n = 0) \in \text{LHP} \]

- How can we determine that? Two methods:

  (1) Direct factorization, Matlab, algebra:
  \[ a_0 s^n + a_1 s^{n-1} + \cdots + a_n = K(s - p_1)(s - p_2) \cdots (s - p_n) = 0 \]
  - That cannot be done on hands (often), need a computer

  (2) Routh’s Stability Criterion:
  - for any polynomial of any degree, determine \( \# \) of roots in the LHP, RHP, or \( j\omega \) axis without having to solve the polynomial
  - Advantages: Less computations + gives discrete answers
So, the RHSC only tells me whether a polynomial (denominator of a TF) has roots in LHP, RHP, or $j\omega$ axis, not the exact locations, which answers stability question of control systems

The opposite is not always true!

How does this work:

- First, if $a_0s^n + a_1s^{n-1} + \cdots + a_n$ is stable, then $a_0, a_1, \cdots, a_n$ have the same sign and are nonzero
- Examples: $(s^2 - s + 1)$ is not stable, $s^4 + s^3 + s^2 + 1$ is not stable
- $s^4 + s^3 + s^2 + s + 1$ is undetermined
How to Apply the RHSC?

- **Objective:** given $a_0s^n + a_1s^{n-1} + \cdots + a_n \Rightarrow$ determine if polynomial is stable

**Step 1** Determine if all coefficients of $a_0s^n + a_1s^{n-1} + \cdots + a_n$ have the same sign & nonzero

**Step 2** If the answer to Step 1 is NO, then system is unstable

**Step 3** Arrange all the coefficients in this *Routh-Array* format:

\[
\begin{array}{cccccc}
\text{s}^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
\text{s}^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
\text{s}^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
\text{s}^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & & & & & \\
\text{s}^2 & e_1 & e_2 & & & \\
\text{s}^1 & f_1 & & & & \\
\text{s}^0 & g_1 & & & & \\
\end{array}
\]

\[
\begin{align*}
b_1 &= \frac{a_1a_2 - a_0a_3}{a_1} & b_2 &= \frac{a_1a_4 - a_0a_5}{a_1} \\
c_1 &= \frac{b_1a_3 - a_1b_2}{b_1} & c_2 &= \frac{b_1a_5 - a_1b_3}{b_1} \\
\end{align*}
\]
(Step 4) \# RHP roots = \# of sign changes in the first column

(Step 5) Stability determination: \( a_0 s^n + a_1 s^{n-1} + \cdots + a_n \) is stable if the first column has no sign change
## RHSC Example — 1

- Determine the stability of:

\[ s^4 + 2s^3 + 3s^2 + 4s + 5 = 0 \]

- Apply the RHSC:

\[
\begin{array}{c|ccc}
   s^4 & 1 & 3 & 5 \\
   s^3 & 2 & 4 & 0 \\
   s^2 & \frac{2 \cdot 3 - 4 \cdot 1}{2} = 1 & \frac{2 \cdot 5 - 1 \cdot 0}{2} = 5 \\
   s^1 & \frac{1 \cdot 4 - 2 \cdot 5}{1} = -6 \\
   s^0 & = ? \\
\end{array}
\]

(S. 4–5) # RHP roots = # of sign changes = 2 ⇒ two RHP roots ⇒ unstable polynomial
RHSC Example — 2

What is a condition on $a_0, a_1, a_2, a_3$ such that the polynomial is stable (all are +ve)?

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

Apply the RHSC:

\[
\begin{array}{c|cc}
 s^3 & a_0 & a_2 \\
 s^2 & a_1 & a_3 \\
 s^1 & \frac{a_1 \cdot a_2 - a_0 \cdot a_3}{a_1} & \\
 s^0 & a_3 & \\
\end{array}
\]

(S. 4–5) Need no sign change in the first column $\Rightarrow$ need $a_1 a_2 > a_0 a_3$, since $a_i > 0 \forall i$
Given the above unity-feedback system, and

\[ G(s) = \frac{K}{s(s^2 + 10s + 20)} \]

find range of \( K \) s.t. the CLTF is stable

**Solution:** first, find CLTF; \( H(s) = \frac{K}{s^3 + 10s^2 + 20s + K} \)

- Apply the RHSC: Steps 1 and 2; \( K > 0 \) and:

\[
\begin{array}{c|cc}
 s^3 & 1 & 20 \\
 s^2 & 10 & K \\
 s^1 & -\frac{1}{10}(K - 200) \\\n s^0 & K \\
\end{array}
\]

(S. 4–5) Need no sign change in the first column \( \Rightarrow \) need \( K < 200 \) and \( K > 0 \), \( \Rightarrow \) \( 0 < 200 < K \)
Special Case 1

- **Sign of 0?** What if 1 of the entries in the first column is 0?
- **Solution:** replace 0 with $\epsilon$, where $\epsilon$ is a small +ve number
- **Case 1:** if the sign of the coefficient above the zero ($\epsilon$) is the same as the sign under $\epsilon \Rightarrow$ there are pair of complex roots
  - **Example:** $s^3 + 2s^2 + s + 2 = 0$

```
<table>
<thead>
<tr>
<th>s^3</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>s^1</td>
<td>0 $\approx \epsilon$</td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
```

- **Case 2:** if the sign of the coefficients above and below $\epsilon$ change $\Rightarrow$ there is a sign change $\Rightarrow$ apply Step 5
  - **Example:** $s^3 - 3s + 2 = (s - 1)^2(s + 2) = 0$

```
<table>
<thead>
<tr>
<th>s^3</th>
<th>1</th>
<th>-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>s^2</td>
<td>0 $\approx \epsilon$</td>
<td>2</td>
</tr>
<tr>
<td>s^1</td>
<td>$-3 - \frac{2}{\epsilon}$</td>
<td></td>
</tr>
<tr>
<td>s^0</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
```
Special Case 2 + Example

- What if an entire row is zero? Then we have:
  - (a) two real roots with equal magnitudes and opposite signs and/or
  - (b) two complex conjugate roots
- Solution illustrated with this example:
  - Example: \( p(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12 = 0 \)

| \( s^5 \) | 1 | 11 | 28 |
| \( s^4 \) | 5 | 23 | 12 |
| \( s^3 \) | 6.4 | 25.6 |
| \( s^2 \) | 3 | 12 |
| \( s^1 \) | 0 | 0 | \( \text{old row, define aux. polynomial: } P(s) = 3s^2 + 12 \) |
| \( s^1 \) | 6 | 0 | \( \text{new row, define aux. polynomial: } P'(s) = 6s + 0 \) |
| \( s^0 \) | 12 |

(Step 4) Find roots of auxiliary polynomial: \( 3s^2 + 12 = 0 \Rightarrow p_{1,2} = \pm j2 \)
(Step 5) \( p_{1,2} \) are both roots for the original polynomial
(Step 6) Count sign changes: none, hence no additional RHP roots
Another Example

**Example:** \( p(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0 \)

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^5 )</td>
<td>1</td>
<td>24</td>
</tr>
<tr>
<td>( s^4 )</td>
<td>2</td>
<td>48</td>
</tr>
<tr>
<td>( s^3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s^2 )</td>
<td>8</td>
<td>96</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>24</td>
<td>-50</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>112.7</td>
<td>0</td>
</tr>
</tbody>
</table>

(Step 4) Find roots of auxiliary polynomial:

\[ 2s^4 + 48s^2 - 50 = 0 \Rightarrow \ p_{1,2,3,4} = \pm j5, \pm 1 \]

(Step 5) \( p_3 \) in RHP, then at least one RHP pole

(Step 6) Count sign changes: once, hence one more additional RHP root

**Conclusion:** one RHP pole — verification:

\[ p(s) = (s + 1)(s - 1)(s + j5)(s - j5)(s + 2) = 0 \]
Tracking Error

- What is tracking? Why is tracking important?
  - Tracking is an important task in control systems
    * Objective: track a certain reference signal \((reference(t) \text{ or } u(t))\)
    - Often, \(ref.(t)\) is a step function or piecewise constant signals
  - Tracking is typically achieved via unity-feedback control systems
  - **Definition 1:** tracking error \(e(t) = u(t) - y(t)\)
  - **Definition 2:** steady-state error (SSE) \(e_{ss} = e(\infty)\)

Wait, we can apply FVT here \(\Rightarrow e_{ss} = \lim_{s \to 0} sE(s)\)

**Important point:** SSE only defined if system is stable

**Target:** study SSE for a unity-feedback system
What Inputs Can We Consider?

Unit step input: \( u(t) = 1, \quad t \geq 0 \quad \Rightarrow U(s) = \frac{1}{s} \)

Unit ramp input: \( u(t) = t, \quad t \geq 0 \quad \Rightarrow U(s) = \frac{1}{s^2} \)

Unit acceleration input: \( u(t) = \frac{t^2}{2}, \quad t \geq 0 \quad \Rightarrow U(s) = \frac{1}{s^3} \)

In general: \( u(t) = \frac{t^k}{k!}, \quad t \geq 0 \quad \Rightarrow U(s) = \frac{1}{s^{k+1}} \)

- Many system inputs can be approximated with scaled polynomials
- If your system can track high order inputs (e.g., \( u(t) = t^{10} + 5t^4 - 7 \)), then your system has an excellent ability in tracking **arbitrary inputs**
System Type (More Definitions)

A unity-feedback system with an OLTTF

\[ G(s) = \frac{K(T_a s + 1) \cdots (T_m s + 1)}{s^N(T_b s + 1) \cdots (T_n s + 1)} \]

is called type N where N is the # of poles of G(s) at s = 0

- Examples

- **Goal**: find SSE for different system types & test inputs (unit step, impulse, ramp)
SSE for a Unit-Step Input

\[ e_{ss} = \lim_{s \to 0} sE(s), \text{ if system is stable} \]

- We now want to find \( e_{ss} \) for any given \( G(s) \).
- Recall (from Module 04 and Exam 1) that \( \frac{E(s)}{U(s)} = \frac{1}{1 + G(s)} \).
- Then, what's \( e_{ss} = e(\infty) \) if \( u(t) = 1 \)?
  - **Answer:** \( e_{ss} = \frac{1}{1 + K_p} \), \( K_p = \lim_{s \to 0} G(s) \).
  - \( K_p \) is called the static position error constant.
  - What would \( e_{ss} \) for Type 0 systems? Type 1?
  - **Answer:** Type 0, it's constant (above), Types 1 and above, it's 0.
- **Conclusion 1:** Type 0 systems track unit step with finite SSE.
- **Conclusion 2:** Type 1 or higher systems track unit step with 0 SSE.
SSE for a Unit-Step Input

\[ e_{ss} = \lim_{s \to 0} sE(s) \quad , \quad \frac{E(s)}{U(s)} = \frac{1}{1 + G(s)} \]

- Then, what’s \( e_{ss} = e(\infty) \) if \( u(t) = t \)?

- **Answer:** \( e_{ss} = \frac{1}{K_v} \), \( K_v = \lim_{s \to 0} sG(s) \)

- \( K_v \) is called the static velocity error constant

- What would \( e_{ss} \) for Type 0 systems? Type 1?

- **Answer:** Type 0, it’s infinity! Why?

- **Conclusion 1:** Type 0 systems **cannot track unit ramp input**

- **Conclusion 2:** Type 1 systems track unit ramp step with finite SSE

- **Conclusion 3:** Type 2 or higher systems track unit ramp unit step with 0 SSE
## Summary of the Results

<table>
<thead>
<tr>
<th></th>
<th>Unit step input ( u(t)=1 )</th>
<th>Unit ramp input ( u(t)=t )</th>
<th>Acceleration input ( u(t)=t^2/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type 0 systems</strong></td>
<td>( \frac{1}{1+K_p} )</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
<tr>
<td></td>
<td>( K_p = G(0) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Type 1 systems</strong></td>
<td>0</td>
<td>( \frac{1}{K_v} )</td>
<td>( \infty )</td>
</tr>
<tr>
<td></td>
<td>( K_v = \lim_{s \to 0} sG(s) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Type 2 systems</strong></td>
<td>0</td>
<td>0</td>
<td>( \frac{1}{K_a} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( K_a = \lim_{s \to 0} s^2G(s) )</td>
</tr>
</tbody>
</table>

- You should not memorize any of these results — you should be able to derive all of these 9 results.
- Before you compute anything, verify that the system is stable.
Design Example 1

For the above given system, and assuming that \( u(t) = 1 \), find \( K \) such that the SSE is as small as possible

**Answer:**
Assume that \( u(t) = t \), find \( K \) such that the SSE is zero

**Answer:** First, find the overall transfer function:

\[
H(s) = \frac{C(s)}{R(s)} = (1 + ks) \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

Now, find \( E(s) \) then \( e_{ss} \) via FVT

\[
E(s) = R(s) - C(s) = \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) R(s) = \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \frac{1}{s^2}
\]

\[
\Rightarrow e_{ss} = e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \left(\frac{s^2 + 2\zeta\omega_n s - \omega_n^2 ks}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right) \frac{1}{s^2} = 2\zeta\omega_n - \omega_n^2 k
\]

We want \( e_{ss} = 0 \) \( \Rightarrow \) set \( k = \frac{2\zeta}{\omega_n} \) to achieve that
Design Example 3

For the above given system, and assuming that

\[ G(s) = \frac{K}{s^3 + s^2 + 2s - 4}, \]

obtain the SSE for unit step input when \( K = 1, 5, \text{ or } 10. \)

(1) First, we have to find the range for \( K \) s.t. system (CLTF) is stable

(2) Routh-Array for \( s^3 + s^2 + 2s + K - 4 = 0: \)

| \( s^3 \) | 1 | 2 |
| \( s^2 \) | 1 | \( K - 4 \) |
| \( s^1 \) | 6 - \( K \) |
| \( s^0 \) | \( K - 4 \) |

\[ \Rightarrow \text{system is stable if } 4 < K < 6 \]

(3) \( \therefore \text{for } K = 1, 10, \text{ SSE doesn’t exist. System is Type 0 } \Rightarrow \text{for } K = 5, \)

SSE is: \[ e_{ss} = \frac{1}{1 + G(0)} = -4 \]
Design Example 4

For the above given system, assume that

\[
G(s) = \frac{1}{s^3 + s^2 + 2s - 0.5}, \quad C(s) = 1 + \frac{K}{s}.
\]

For \( K \geq 0 \), obtain the range of \( K \) such that the CLTF is stable

Do this problem at home

**Solution:** \( 0 < K < 0.75 \)
Course Progress

**Modeling (5-6 Weeks)**
- Laplace Transforms
- Transfer Functions
- Solution of ODEs
- Modeling of Systems
- Block Diagrams
- Linearization

**Analysis (7-8 Weeks)**
- 1st & 2nd Order Systems
  - Time Response
  - Transient & Steady State
  - Frequency Response
  - Bode Plots
  - RH Criterion
  - Stability Analysis

**Design (5-6 Weeks)**
- Root-Locus
- Modern Control
- State-Space
- MIMO System Properties
Questions And Suggestions?

Any questions?

Thank You!

Please visit engineering.utsa.edu/~taha

IFF you want to know more 😊