Module 03
Modeling of Dynamical Systems

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EE 3413: Analysis and Design of Control Systems

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Module 3 Outline

1. Physical laws and equations
2. Transfer function model
3. Model of electrical systems
4. Model of mechanical systems
5. Examples
   - Reading material: Dorf & Bishop, Section 2.3
By definition, dynamical systems are dynamic because they change with time.

Change in the sense that their intrinsic properties evolve, vary.

Examples: coordinates of a drone, speed of a car, body temperature, concentrations of chemicals in a centrifuge.

Physicists and engineers like to represent dynamic systems with equations.

Why? Well, the answer is fairly straightforward.

Dynamic model often means a differential equations.
Physical Laws

- For many systems, it’s easy to understand the physics, and hence the math behind the physics
  - Examples: circuits, motion of a cart, pendulum, suspension system
- For the majority of dynamical systems, the actual physics is complex
- Hence, it can be hard to depict the dynamics with ODEs
  - Examples: human body temperature, thermodynamics, spacecrafts
- This illustrates the needs for *models*
- **Dynamic system model**: a mathematical description of the actual physics
What are Transfer Functions?

\[ u(t) \quad \rightarrow \quad y(t) \]

* **TFs**: a mathematical representation to describe relationship between inputs and outputs of the physics of a system, i.e., of the differential equations that govern the motion of bodies, for example

- **Input**: always defined as \( u(t) \)—called control action
- **Output**: always defined as \( y(t) \)—called measurement or sensor data
- TF relates the derivatives of \( u(t) \) and \( y(t) \)
- Why is that important? Well, think of \( \sum F = ma \)
- \( F \) above is the input (exerted forces), and the output is the acceleration, \( a \)
- Give me the equations, please...
Construction of Transfer Functions

For linear systems, we can often represent the system dynamics through an $n$th order ordinary differential equation (ODE):

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + a_{n-2}y^{(n-2)}(t) + \cdots + a_0y(t) =$$

$$u^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + b_{m-2}u^{(m-2)}(t) + \cdots + b_0u(t)$$

The $y^{(k)}$ notation means we’re taking the $k$th derivative of $y(t)$.

Typically, $m > n$.

Given that ODE description, we can take the LT (assuming zero initial conditions for all signals):

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$
What are Transfer Functions?

Given this TF:

\[
H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}
\]

For a given control signal \( u(t) \) or \( U(s) \), we can find the output of the system, \( y(t) \), or \( Y(s) \).

**Impulse response:** defined as \( h(t) \)—the output \( y(t) \) if the input \( u(t) = \delta(t) \)

**Step response:** the output \( y(t) \) if the input \( u(t) = 1^+(t) \)

For any input \( u(t) \), the output is: \( y(t) = h(t) * u(t) \)

But...Convolutions are nasty...Who likes them?
So, we can take the Laplace transform: 

\[ Y(s) = H(s)U(s) \]

Typically, we can write the TF as:

\[
H(s) = \frac{Y(s)}{U(s)} = \frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}
\]

Roots of numerator are called the **zeros** of \( H(s) \) or the system

Roots of the denominator are called the **poles** of \( H(s) \)

Pole zero plot:

- \( \circ \): zero
- \( \times \): pole
Example

Given: \( H(s) = \frac{2s + 1}{s^3 - 4s^2 + 6s - 4} \)

- **Zeros**: \( z_1 = -0.5 \)
- **Poles**: solve \( s^3 - 4s^2 + 6s - 4 = 0 \), use MATLAB's `roots` command
  
  \[ \text{poles}=\text{roots}[1\ -4\ 6\ -4]; \text{poles} = \{2, 1 + j, 1 - j\} \]

- **Factored form**:
  \[
  H(s) = 2 \frac{s + 0.5}{(s - 2)(s - 1 - j)(s - 1 + j)}
  \]
Analyzing Generic Physical Systems

**Seven-step algorithm:**

1. Identify dynamic variables, inputs \((u)\), and system outputs \((y)\)
2. Focus on one component, analyze the dynamics (physics) of this component
   - How? Use Newton’s Equations, KVL, or thermodynamics laws...
3. After that, obtain an \(n\)th order ODE:
   \[
   \sum_{i=1}^{n} \alpha_i y^{(i)}(t) = \sum_{j=1}^{m} \beta_j u^{(j)}(t)
   \]
4. Take the Laplace transform of that ODE
5. Combine the equations to eliminate internal variables
6. Write the transfer function from input to output
7. For a certain control \(U(s)\), find \(Y(s)\), then \(y(t) = \mathcal{L}^{-1}[Y(s)]\)
Active Suspension Model

- Each car has 4 active suspension systems (on each wheel)
- System is nonlinear, but we consider approximation. **Objective?**
- **Input:** road altitude $r(t)$ (or $u(t)$), **Output:** car body height $y(t)$
We only consider one of the four systems

System has many components, most important ones are: body \( (m_2) \) & wheel \( (m_1) \) weights

By Newton’s Second Law

\[
m_1 \ddot{x} = k_s(y - x) + b(y - \dot{x}) - k_w(x - r)
\]

\[k_s(y - x) \quad b(y - \dot{x})\]

\[k_w(x - r)\]
We now have 2 equations depicting the car body and wheel motion:

Objective: find the TF relating output \( y(t) \) to input \( r(t) \)

What is \( H(s) = \frac{Y(s)}{R(s)} \)?
Active Suspension Model — Transfer Function

• Differential equations (in time):

\[ m_1 \ddot{x}(t) = k_s(y(t) - x(t)) + b(\dot{y}(t) - \dot{x}(t)) - k_w(x(t) - r(t)) \]
\[ m_2 \ddot{y}(t) = -k_s(y(t) - x(t)) - b(\dot{y}(t) - \dot{x}(t)) \]

• Take Laplace transform given zero ICs:

– Solution:

• Find \( H(s) = \frac{Y(s)}{R(s)} \)

– Solution:
Basic Circuits Components

resistor
\[ v(t) = Ri(t) \]
\[ V(s) = RI(s) \Rightarrow \frac{V(s)}{I(s)} = R \]

inductor
\[ v(t) = L \frac{di(t)}{dt} \]
\[ V(s) = LsI(s) \Rightarrow \frac{V(s)}{I(s)} = Ls \]

capacitor
\[ i(t) = C \frac{dv(t)}{dt} \]
\[ I(s) = CsV(s) \Rightarrow \frac{V(s)}{I(s)} = \frac{1}{Cs} \]
Basic Circuits — RLCs & Op-Amps

\[ v_i(t) : \text{input} \]
\[ v_o(t) : \text{output} \]

Transfer function \[ \frac{V_o(s)}{V_i(s)} \]

\[ v_i(t) : \text{input} \]
\[ v_o(t) : \text{output} \]

Transfer function \[ \frac{V_o(s)}{V_i(s)} \]
TF of an RLC Circuit — Example

Objective: Find TF

\[ v_i(t) : \text{input} \]
\[ v_o(t) : \text{output} \]

Transfer function \( \frac{V_o(s)}{V_i(s)} \)

- Apply KVL (assume zero ICs):

\[ v_i(t) = Ri(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int i(\tau) d\tau \]

\[ v_o(t) = \frac{1}{C} \int i(\tau) d\tau \]

- Take LT for the above differential equations:

\[ V_i(s) = RI(s) + LsI(s) + \frac{1}{Cs} I(s) \]

\[ V_o(s) = \frac{1}{Cs} I(s) \Rightarrow I(s) = CsV_o(s) \]

\[ \Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + RCs + 1} \]
s-Domain Circuit Analysis

Time domain (t domain)

- Linear Circuit
- Differential equation
- Classical techniques
- Response waveform

Complex frequency domain (s domain)

- Laplace Transform $\mathcal{L}$
- Laplace Transform $\mathcal{L}$
- Inverse Transform $\mathcal{L}^{-1}$

- Transformed Circuit
- Algebraic equation
- Algebraic techniques
- Response transform
Dynamic Models in Nature

- Predator-prey equations are 1st order non-linear, ODEs
- Describe the dynamics of biological systems where 2 species interact
- One species as a predator and the other as prey
- Populations change through time according to these equations:

\[
\begin{align*}
\dot{x}(t) &= \alpha x(t) - \beta x(t)y(t) \\
\dot{y}(t) &= \delta x(t)y(t) - \gamma y(t)
\end{align*}
\]

- \(x(t)\): # of preys (e.g., rabbits)
- \(y(t)\): # of predators (e.g., foxes)
- \(\dot{x}(t), \dot{y}(t)\): growth rates of the 2 species—function of time, \(t\)
- \(\alpha, \beta, \gamma, \delta\): +ve real parameters depicting the interaction of the species
Mathematical Model

\[ \frac{dx(t)}{dt} = \alpha x(t) - \beta x(t)y(t) \]
\[ \frac{dy(t)}{dt} = \delta x(t)y(t) - \gamma y(t) \]

- Prey's population grows exponentially \((\alpha x(t))\)—why?
- Rate of predation is assumed to be proportional to the rate at which the predators and the prey meet \((\beta x(t)y(t))\)
- If either \(x(t)\) or \(y(t)\) is zero then there can be no predation
- \(\delta x(t)y(t)\) represents the growth of the predator population
- No prey \(\Rightarrow\) no food for the predator \(\Rightarrow\) \(y(t)\) decays
- Is there an equilibrium? What is it?
Nonlinear Dynamical Systems

- Let’s face it: most dynamical systems are **nonlinear**
- Nonlinearities can be seen in the ODEs, e.g.:
  \[
  \dot{y}(t) + \dot{y}(t)\ddot{y}(t) + \cos(y(t)) = 2u(t) + \arctan(e^{\cos(u(t))})
  \]
- Examples: electromechanical systems, electronics, hydraulic systems, thermal, etc...
- **Why do we hate nonlinear systems?**
  - Well, because we cannot solve ODEs tractably if they are not linear
  - I mean we can, but they’re hard—and remember, we’re lazy
- **Solution: linearize** nonlinear equations
- Btw…most nonlinear systems are linear for a **short period of time**
- So, it’s legit to linearize for a **short period of time**
Linearization — The Main Idea

- Linearization is one of the most important techniques in control theory.
- Without it, all our analysis of nonlinear systems becomes pointless.
- First, let's assume that a nonlinear system is linearized around an operating point.
- Operating point is often called equilibrium point.
- Main idea:

\[ f(x_0 + \delta x) \]

\[ f(x) \]  

Operating point \( x_0 \)  

Old coordinate  

Nonlinear function  

Linear approximation  

\( \delta x \)  

New coordinate
Nonlinear equation (or system): \( \dot{x}(t) = f(x, u) \)

**Equilibrium points**: \( u_e, x_e \)

Equilibrium **deviation**: \( \delta u(t) = u(t) - u_e, \delta x(t) = x(t) - x_e \)

Taylor series expansion around \( u_e, x_e \):

\[
\dot{x}(t) \approx f(x_e, u_e) + (\delta x(t)) \left. \frac{\partial f(x, u)}{\partial x} \right|_{x_e, u_e} + (\delta u(t)) \left. \frac{\partial f(x, u)}{\partial u} \right|_{x_e, u_e}
\]

Hence:

\[
\delta \dot{x}(t) \approx (\delta x(t)) \left. \frac{\partial f(x, u)}{\partial x} \right|_{x_e, u_e} + (\delta u(t)) \left. \frac{\partial f(x, u)}{\partial u} \right|_{x_e, u_e}
\]

This relationship is a linear one between \( \delta x \) and \( \delta u \)
Linearization — Example

Pendulum motion:

\[ f(x, u) = -\frac{g}{L} \sin(x(t)) + \frac{1}{mL^2} u(t) \]

- \( x(t) \): angle (\( \theta \)), \( u(t) \): force

- Given equilibrium points: \( u_e = 0, x_e = \pi \)

- Taylor series expansion around 0, \( \pi \):

\[ \delta f(\delta x, \delta u) \approx \frac{g}{L} \delta x(t) + \frac{1}{mL^2} \delta u(t) \]

- This relationship is a linear one between \( \delta x \) and \( \delta u \): only valid in the vicinity of the equilibrium point
Roadmap Revisited

**Modeling** (5-6 Weeks)
- Laplace Transforms
- Transfer Functions
- Solution of ODEs
- Modeling of Systems
- Block Diagrams
- Linearization

**Analysis** (7-8 Weeks)
- 1st & 2nd Order Systems
  - Time Response
  - Transient & Steady State
- Frequency Response
- Bode Plots
- RH Criterion
- Stability Analysis

**Design** (5-6 Weeks)
- Root-Locus
- Modern Control
- State-Space
- MIMO System Properties
Questions And Suggestions?

Any questions?

Thank You!

Please visit engineering.utsa.edu/~taha

IFF you want to know more 😊