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EE-3424, Mathematics in Signals and Systems

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CONTENTS

PART I: LECTURES

P R E F A C E

Digital signal processing (DSP) is an area of science and engineering that has developed rapidly over the past thirty years. The rapid developed DSP is a result of the significant advances in digital computer technology and integrated-circuit fabrication. These lecture-notes are based on the instructor notes, the material of many text-books, and among them, the following books should be mentioned:

1. *Signals, Systems, and Transforms*, (5th Ed., 2014), by Charles L. Philips, John M. Parr, and Eve A. Riskin, Chapters 7, 8, 10-13.
2. *Signals and Systems: Analysis using Transform methods and MATLAB* (2nd Ed., 2004) by M.J. Roberts.
3. *Fundamentals of Signals and Systems: with MATLAB Examples* (2000) by Edward Kamen and Bonnie Heck.
4. *Signals and Systems*, (2nd Ed., 2002) by Simon Haykin and Barry Van Veen.
5. ...

I. Introduction in Mathematics in Signals and Systems

Signal is a real or abstract concept of what carries, represents, or encodes information. Signals can be manipulated, stored, or transmitted by a physical process. Examples: *speech signals, audio signals, video signals, images, radar signals, biomedical signals.* System is a transformation or a more complicated process of transformation of signals, which may include the recording, changing, and transmitting signals.

Examples: *1. Audio compact disk (CD) records music signals on the disk as a sequence of numbers. 2. A system for converting these numbers to an acoustic signal is the CD player.*

Mathematics is an appropriate language for describing and understanding many signals and systems. Mathematical equations are used to describe and represent signals and systems and to design new systems to achieve desired results. Examples: *differential equations of linear systems, Fourier transformation, filters used in signal and image denoising, restoration, and enhancement.*

Signal processing plays a central role in modern sciences and technology. Applications are founded everywhere, including speech communication, acoustic, biomedical engineering, seismology, and many others. The theory of signal processing is concerned in representation, transmission, and manipulations of signals. Until to the 1960s the technology for signal processing has been carried in general by analog methods. The evolution of digital computer and microprocessors has been pushed the developing discrete versions of signal processing methods. Digital signal processing becomes applicable in many areas beginning from medical imaging to radar processing. That growth has created a massive amount of data, which is important to analyze, process, and transmit signals. The digital signal and image processing have successful applications in geology, biology, meteorology, astronomy, radio-location, television, medical diagnosis, and many others domains in science and engineering.

A. Modeling and Signals

We consider two topics of signals and systems as related to engineering:

1. Modeling of *physical systems* by mathematical equations.
2. Modeling of *physical signals* by mathematical functions.

Model: It is advantageous to represent a device or an entire system by a circuit model. For instance, we can consider the 10m of copper wire is wound into the form of a multi-turn coil and that a voltage of variable frequency is applied. If the ratio of applied voltage V to resulting current I is measured as a function of frequency. The model of this simple physical system is described by the equation

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{\tau<t} i(\tau)d\tau = v(t)$$

where R is the parameter of resistance, C is capacitance, and L is induction.

Signals. Our word is full of signals, both natural and man-made. A signal can be defined as a function that conveys information about the state or behavior of physical system. Signals are represented mathematically as functions of one or more independent variables.

For example, the functions $f(t) = 2t + 1$ and $f(t) = 3t^2 + 2t + 1$ describe two signals, which vary respectively linearly and quadratically with variable t.

Example of signals:

– The daily highs and lows in temperature. We can express the temperature at a designated point in the room as a function $\theta(x, y, z, t)$ of four independent variables, where (x, y, z) are space coordinates of the point and t is time.

– The periodic electrical signals generated by the heart. Electrocardiogram (ECG) signal provides information about the activity of the patient's heart. Electroencephalogram (EEG) signal provides information about the activity of the brain. The variation is air pressure when we speak. A speech can be represented mathematically as function of time.

– The music we hear from compact disk (CD) player, which is due to changes in the air pressure caused by the vibration of the speaker diaphragm. The information stored on CD is in a digital form and, then, is converted to the analog form before we can hear the music.

– Seismic signal is shown in Fig. 1.

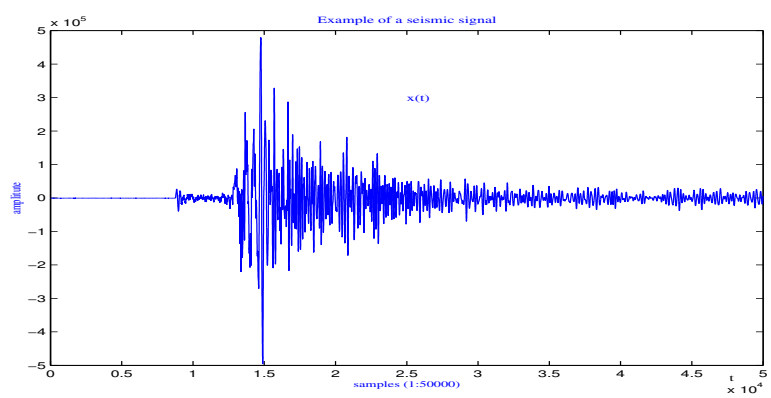


Fig. 1. Seismic signal.

– A photograph image (2-D signal) is represented as a brightness function(s) of two spatial variables (coordinates in a plane). As an example, Figure 2 shows 506 random balls placed randomly in the 3-D box $640 \times 520 \times 100$. At each point (x, y, z) of those balls, an intensity $\theta(x, y, z)$ is defined that is concentrated in the center of balls (see Fig. 3).

The projection of the intensity along the middle lines of Y-axis and X-axis are shown respectively on Figs. 4(a) and (b).

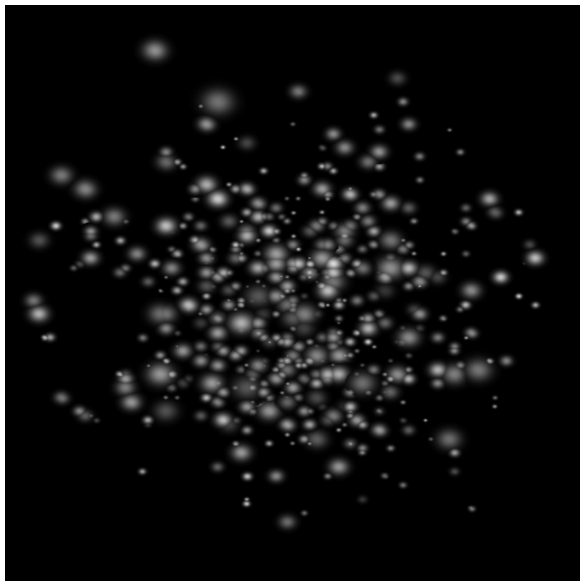


Fig. 2. Image with random balls.

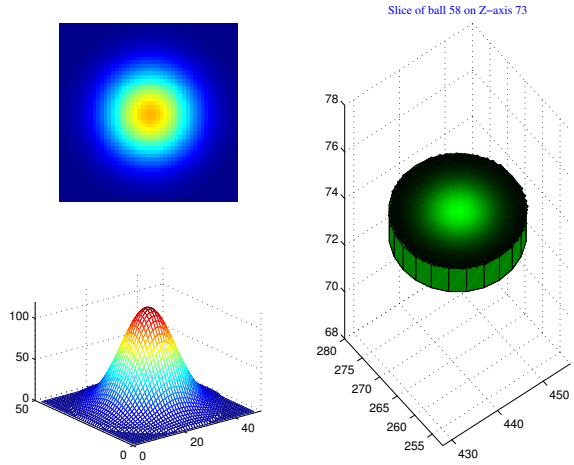


Fig. 3. Random balls.

II. Elementary Continuous-time Signals

Signals *convey* information and include physical quantities such as voltage, current, and intensity. We will consider 1-D signals (analog or discrete) that are functions (continuous or step-wise continuous) usually of time or frequency. They are two important variables, time and frequency (t and ω), used in signal processing.

In the one-dimensional case, we used to consider the independent variable of the mathematical representation of a signal as the *time*. The independent variable in the mathematical representation of a signal may be either continuous or discrete.

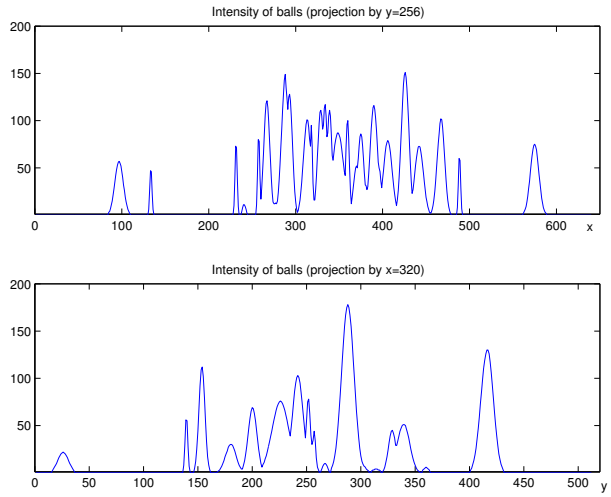


Fig. 4. Projections of the intensity of random balls.

A. Continuous-time and discrete-time signals

Signals may have either a continuous or discrete variable representation, and if certain conditions hold, these representations are entirely equivalent.

Continuous-time signals are often refereed to as analog signals and are defined along a continuum of times. We will notations $x(t), y(t), f(t), \ldots$ for such signals.

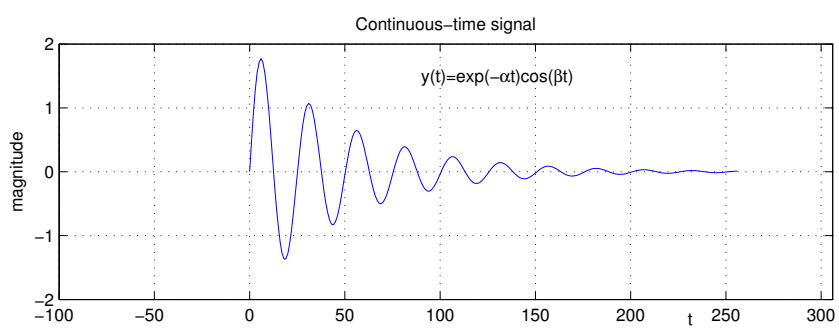


Fig. 5. The signal of finite duration.

Figure 5 shows the graph of a right side signal $x(t)$ with the interval (domain) of definition $[0, 260]$, i.e., when $t \in [0, 260]$. This interval can also be called the time-interval, or time-segment, or continuous-time segment of the signal. The range of the signal are the interval of numbers where the values of the signal lie. In this example, the range is the interval which is smaller than or inside the interval $[-2, 2]$. In general, the range of the signal may be real, i.e., all $x(t)$ are real numbers, or complex, i.e., all $x(t)$ are complex numbers. As a special case, we mention the integer-valued signals, when the range of the signals is a set of integer numbers, for example, numbers $0, 1, 2, \cdots, 255$.

Discrete-time signals are refereed mathematically as a sequences of numbers and those are defined at discrete times (the independent variable has discrete values). We will notations $x[n], y[n], f[n], \ldots,$

or $x_n, y_n, f_n, \ldots$, for such signals. If a discrete signal $y[n]$ was composed (sampled) from a continuous-time signal $y(t)$, i.e., only the values $y(nT)$ were recorded, where T is a small number (it is called *sampling period*), we use to consider this short notation $y[n]$ instead of $y(nT)$.

As an example, Figure 6 shows the original continuous-time signals (modeled by MATLAB) in part (a) and its discrete version in the interval $[0, 128]$, or the discrete signal when the sampling period $T = 1$ in part (b).

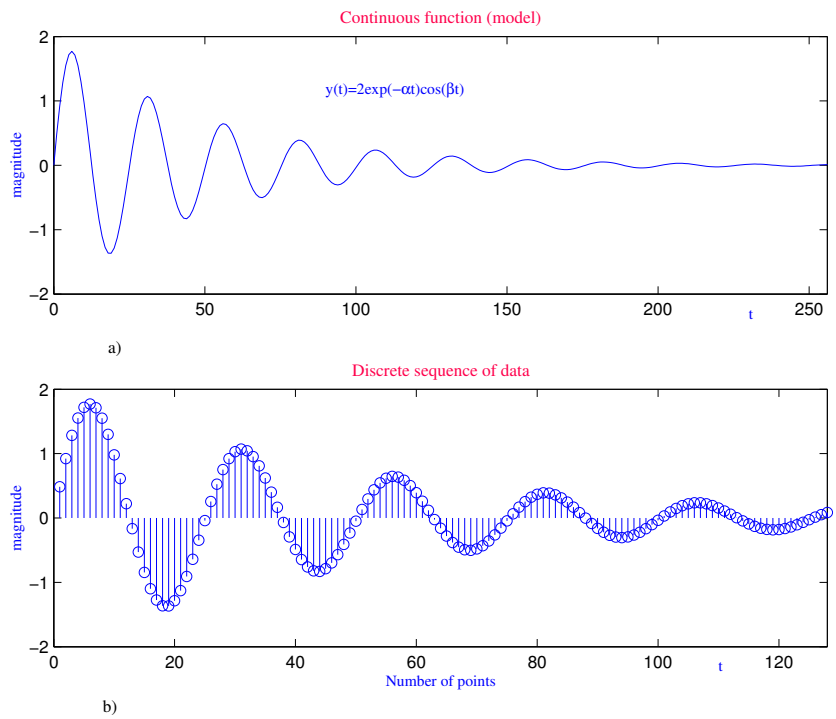


Fig. 6. Graphic representations of sequences: (a) analog signal $y(t) = 2\exp(-\alpha t)\sin(\beta t)$ for $\alpha = 0.02$, $\beta = 0.25$, and (b) discrete analog of the signal, $y[n] = y(nT)$.

Many analog signals can be described by a mathematical expression or graphically by a curve, or by a set of tabulated values. Real signals are not easy to describe quantitatively. They must often be approximated by idealized forms or models. Signals can be of finite or infinite duration. Finite durations signals are called *time-limited*. Signals of semi-infinite extent may be *right-side*, if they are zero for $t < a$ (a is finite), or *left-side*, if they are zero for $t > a$. Signals are *causal* if they are zero for $t < 0$.

A continuous-time signal as a function $x(t)$ may have discontinuity at some points $t_0, t_2, \ldots$. It means that at each of such points, for instance, t_0, the following holds:

$$\lim_{\epsilon \to 0} x(t_0 - \epsilon) \neq \lim_{\epsilon \to 0} x(t_0 + \epsilon), \qquad (\epsilon > 0).$$

As an example, Figure 10 shows the signal which is discontinuous at points $t_0 = -1$ and $t_1 = 1$. Indeed, at point the $t_0 = -1$, we have the following:

$$\lim_{\epsilon \to 0} x(t_0 - \epsilon) = 0 \neq 1 = \lim_{\epsilon \to 0} x(t_0 + \epsilon),$$

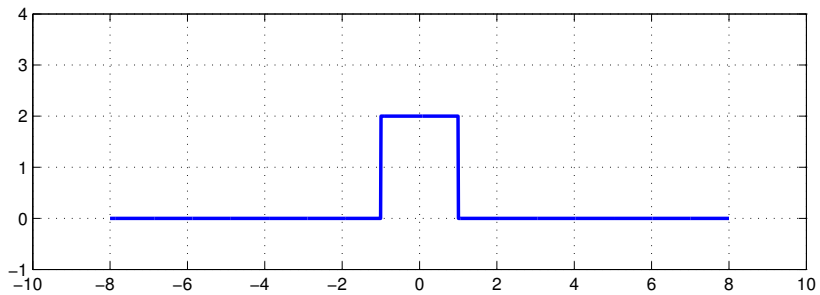


Fig. 7. The rectangle signal $x(t)$ of finite duration.

and at point $t_1 = 1$, we have the following:

$$\lim_{\epsilon \to 0} x(t_1 - \epsilon) = 1 \neq 0 = \lim_{\epsilon \to 0} x(t_1 + \epsilon).$$

A.1 Simple signals

A Linear functions

$$x(t) = at + b, \quad t \in [t_1, t_2],$$

where numbers t_1 and t_2 are boundary points of the domain of the signal $x(t)$. The number a and

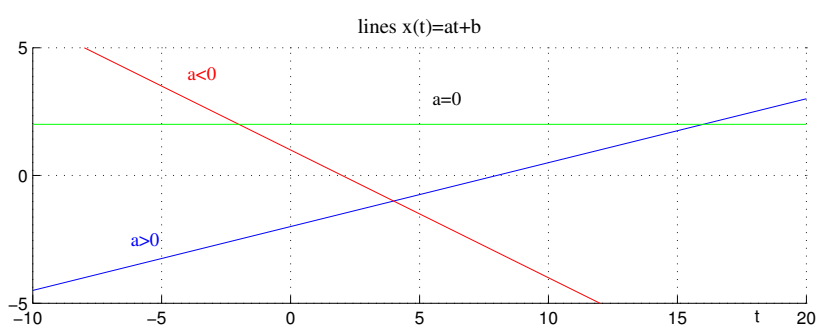


Fig. 8. Functions.

b are two parameters of the line. a defines the slop of the signal and b is the point at which the line intersects with the vertical axis, i.e., $b = x(o)$. We see that, $a = x'(t)$ at each point $t \in [t_1, t_2]$. Figure 8 shows three lines for cases when $a > 0$, a< 0, and $a = 0$. When $a = 0$, the line $x(t) \equiv b$.

Properties:

1. The sum of two different lines $x_1(t) = a_1 t + b_1$ and $x_2(t) = a_2 t + b_2$ is the line

$$x(t) = x_1(t) + x_2(t) = (a_1 + a_2)t + (b_1 + b_2).$$

2. The sum of two different lines $x_1(t) = a_1 t + b_1$ and $x_2(t) = -a_1 t + b_2$ is the line parallel to the horizontal axis

$$x(t) = x_1(t) + x_2(t) \equiv b_1 + b_2.$$

3. Two different lines $x_1(t) = a_1 t + b_1$ and $x_2(t) = a_2 t + b_2$ are parallel, if $a_1 = a_2$ and $b_1 \neq b_2$.

4. The sum of two the lines $x_1(t) = at + b_1$ and $x_2(t) \equiv b_2$ is the line which is parallel to $x_1(t)$,

$$x(t) = x_1(t) + b_2 = at + (b_1 + b_2).$$

B Piece-wise linear signals (functions)
As an example, Figure 9 shows the continuous-time signal which is continuous.

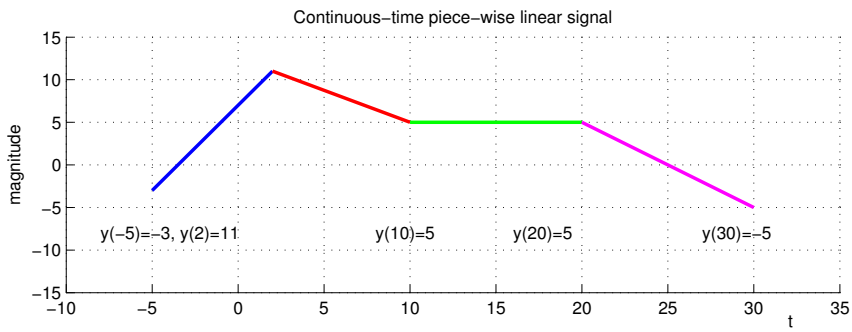


Fig. 9. The signal $y(t)$ of duration of 35 sec.

The time segment of the signal or domain of definition of the signal is the interval $[-5, 35]$. To write analytically this signal, we first consider the equation of the line

$$x(t) = at + b$$

passing two points $(t_1, x(t_1))$ and $(t_3, x(t_2))$:

$$x(t) = \frac{x(t_2) - x(t_1)}{t_2 - t_1} t + \frac{x(t_1)t_2 - x(t_2)t_1}{t_2 - t_1} \tag{1}$$

Here

$$a = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \quad \text{and} \quad b = \frac{x(t_1)t_2 - x(t_2)t_1}{t_2 - t_1}. \tag{2}$$

We now consider and write separately this signal in intervals $[-5, 2]$, $(2, 10]$, $[10, 30)$, and $[30, 35]$, by using Eqs. 1 and 2. This signal can be written in the following form:

$$y(t) = \begin{cases} +\dfrac{11-(-3)}{7} t + \dfrac{-3(2) - 11(-5)}{7}, & \text{if} \quad t \in [-5, 2], \\ -\dfrac{11-5}{8} t + \dfrac{11(10) - 5(2)}{8}, & \text{if} \quad t \in (2, 10], \\ 5, & \text{if} \quad t \in (10, 20], \\ -\dfrac{5-(-5)}{10} t + \dfrac{5(30) - (-5)20}{10}, & \text{if} \quad t \in (20, 30]. \end{cases}$$

Therefore,

$$y(t) = \begin{cases} +2t + 7, & \text{if} \quad t \in [-5, 2], \\ -\dfrac{3}{4} t + \dfrac{25}{2}, & \text{if} \quad t \in (2, 10], \\ 5, & \text{if} \quad t \in (10, 20], \\ -t + 25, & \text{if} \quad t \in (20, 30]. \end{cases}$$

C. Sinusoidal waves (sine and cosine signals)

We consider two functions, which are defined by the following power series:

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \frac{t^{10}}{10!} + \dots$$

and

$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \frac{t^{11}}{11!} + \dots$$

where the factorial of the integer n is defined as $n! = 1 \cdot 2 \cdot 3 \dots \cdot (n-1) \cdot n$ if $n \neq 0$, and $0! = 0$.

These functions are cosine and sine functions which are defined for any time-point t, and in addition

$$-1 \leq \cos(t) \leq 1 \quad \text{and} \quad -1 \leq \sin(t) \leq 1,$$

for any $t \in (-\infty, \infty)$.

Consider the waves defined as

$$x(t) = 2\sin(2t-1) \qquad \text{and} \qquad y(t) = -4\cos(3t+2).$$

The amplitude of the signal $x(t)$ is 2, and for the signal $y(t)$, it is -4. The magnitude of the signal $x(t)$ is 2, and for the signal $y(t)$, it is 4. The first signal oscillates between -2 and 2, and the second signal oscillates between -4 and 4.

Figure 10 shows the signals $x(t)$ and $y(t)$ in parts (a) and (b), respectively.

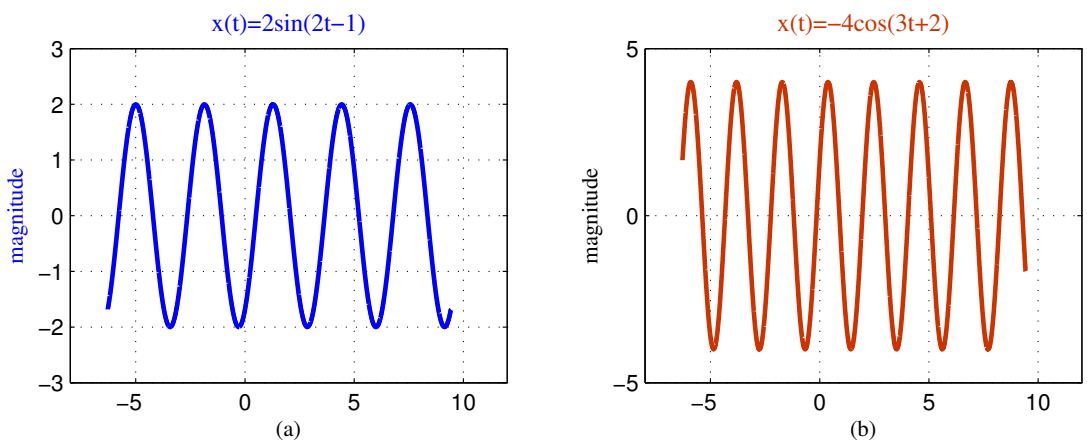


Fig. 10. The graphs of the signals (a) $x(t)$ and (b) $y(t)$ in the time-interval $[-2\pi, 3\pi]$.

The variable t for cosine and sine functions are usually considered as an angle; we will use the letter ϑ instead of t. [1]

The main properties of the trigonometric functions $\sin(\vartheta)$ and $\cos(\vartheta)$:

1. $\sin(\vartheta) = -\cos(\vartheta + \pi/2)$;
2. $\cos(\vartheta) = \sin(\vartheta + \pi/2)$;

[1] ϑ – Greek letter 'vartheta'

3. $\sin(\vartheta + 2\pi k) = \sin(\vartheta)$, for any integer k;
4. $\cos(\vartheta + 2\pi k) = \cos(\vartheta)$, for any integer k;
5. $\cos(\pi k) = (-1)^k = \pm 1$, and $\sin(\pi k) = 0$, when k is integer;
6. $\cos(2\pi(k + 1/2)) = -1$, when k is integer;
7. $\sin(-\vartheta) = -\sin(\vartheta)$;
8. $\cos(-\vartheta) = \cos(\vartheta)$;
9. $\cos(\pi/2) = 0$ and $\sin(0) = 0$;
10. $\cos(0) = 1$ and $\sin(\pi/2) = 1$.

The main equations:

1. $\sin^2(\vartheta) + \cos^2(\vartheta) = 1$;
2. $\sin(2\vartheta) = 2\sin(\vartheta)\cos(\vartheta)$;
3. $\cos(2\vartheta) = \cos^2(\vartheta) - \sin^2(\vartheta) = 2\cos^2(\vartheta) - 1$;
4. $\sin(\vartheta_1 + \vartheta_2) = \sin(\vartheta_1)\cos(\vartheta_2) + \sin(\vartheta_2)\cos(\vartheta_1)$;
5. $\cos(\vartheta_1 + \vartheta_2) = \cos(\vartheta_1)\cos(\vartheta_2) - \sin(\vartheta_2)\sin(\vartheta_1)$;

and a few more with the derivatives:

6. $(\sin(\vartheta))' = \cos(\vartheta)$;
7. $(\cos(\vartheta))' = -\sin(\vartheta)$;
8. $\sin(\vartheta)d\vartheta = -d\cos(\vartheta)$;
9. $\cos(\vartheta)d\vartheta = d\sin(\vartheta)$.

III. Transformations of signals

Let $x(t)$ be a function of one independent variable (time) defined for all values $-\infty < t < +\infty$, i.e., for all $t \in R$, where $R = R^1$ denotes the real line.

A. *Time transformations:*

1. *Time reversal.* Given a signal $x(t)$, a time-reversal transformation of the signal is defined as

$$y(t) = x(-t), \quad -\infty < t < +\infty. \tag{3}$$

Drawing the graph of the signal, we can see that the time-reversal transformation creates the mirror image about the vertical axis.

As an example, Figure 12 shows the signal $x(t)$ defined in the interval $[-1, 2]$ in part a, along with the time-reversal transform in b and c, and two signals together in d. Note that in c, the numbers along the x-axis have been changed only.

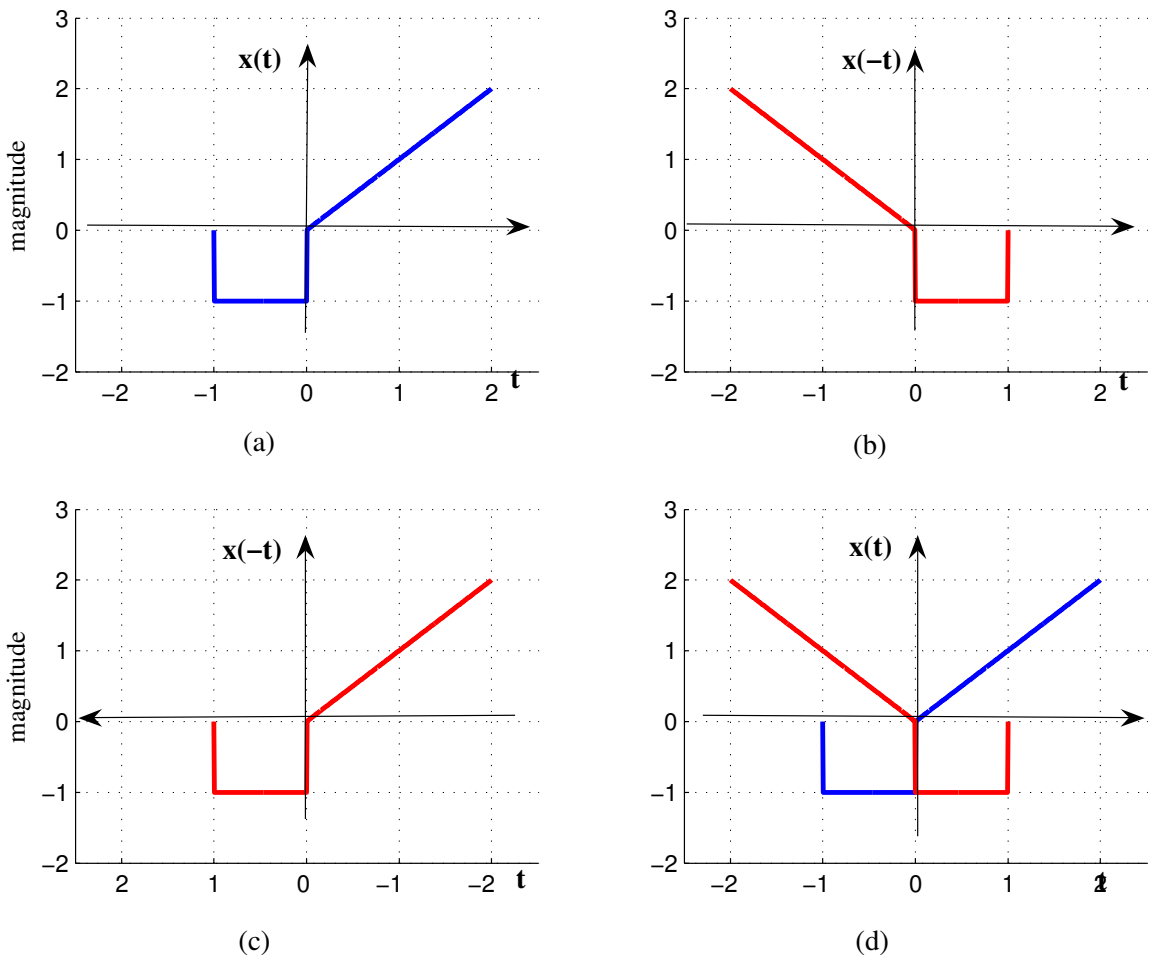


Fig. 12. Signal and its time-reversal transform.

There are many functions which do not change after time-reversal transformation, i.e.,

$$y(t) = x(t), \quad \forall t.$$

For example, such are signals $x(t) = |t|$, $x(t) = t^2$, $x(t) = \cos(x)$, which are called symmetric relative the vertical axis $t = 0$. These symmetric signals are shown in Fig. 13 along with the non symmetric functions t^3 and $\sin(t)$.

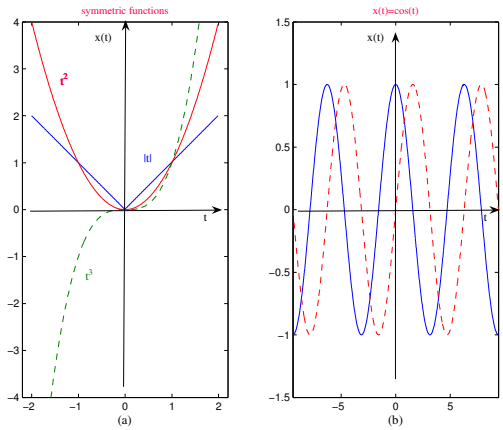


Fig. 13. Symmetric and non symmetric signals.

Note that the above signal $x(t)$ is defined only on the interval $[-1, 2]$. We can thus define and plot $y(t)$ only on the interval $[-2, 1]$, i.e., $y(t)$ is defined by (3) when $t \in [-2, 1]$.

In practice, working with digital signals which are defined (or can be defined) on a finite interval $[a, b]$, we used to determine the time-reversal transformation of the signal relative to the vertical line crossing the middle point of $[a, b]$. In order words, we define the time-reversal transformations as (see Fig. 14)

$$y(t) = x(b + a - t), \quad t \in [a, b]. \tag{4}$$

That results in the mirror image about the vertical axis shifted by $(b + a)/2$ to the right. Indeed, we can write that

$$x(b + a - t) = x\left(\frac{b + a}{2} - \left[t - \frac{b + a}{2}\right]\right).$$

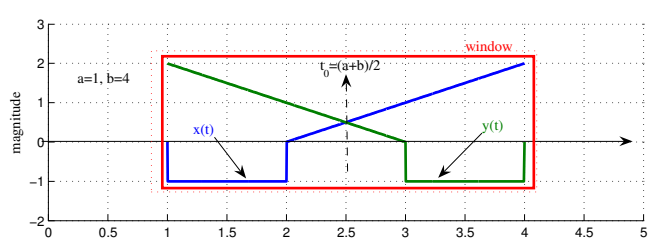


Fig. 14. Time-reversal signals relative to the middle point.

here, we face with a new operation over the function, namely the time-shifting of the signal.

2. *Time shifting.* Given a signal $x(t)$ and a value t_0, a time-shifted version of the signal is defined as

$$y(t) = x_{t_0}(t) = x(t - t_0), \quad -\infty < t < +\infty. \tag{5}$$

When we plot the signal, the time-shifting transformations yields the shift of the vertical axis by value t_0 to the left or right, depending on $t_0 > 0$ or $t_0 < 0$.

As an example, Figure 15 shows the signal $x(t)$ and its shifted transforms by ± 3.5

$$y(t) = x(t - 3.5) \qquad \text{(shifted to the right by 3.5)}$$

and

$$y(t) = x(t + 3.5) \qquad \text{(shifted to the left by 3.5).}$$

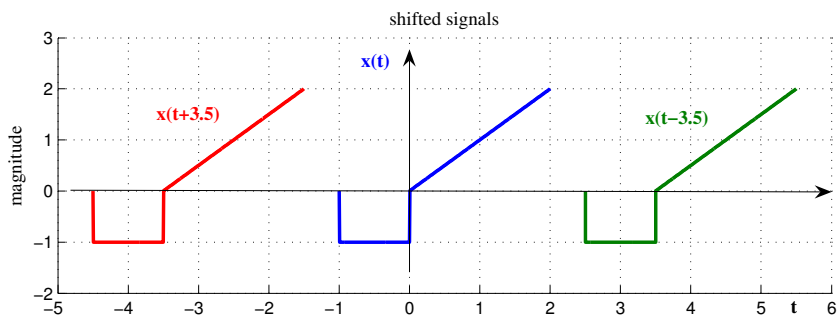


Fig. 15. Signal $x(t)$ and its time-shift transforms by ± 3.5.

Example 1: We consider the cosine signal

$$x(t) = \cos(t), \quad t \in R.$$

Taking values of t_0 equal $\pi/2$ and $-\pi/2$, we obtain respectively (see Fig. 16):

$$\begin{aligned} x_{\pi/2}(t) &= x(t - \pi/2) = \cos(t - \pi/2) = -\sin(t) \\ x_{-\pi/2}(t) &= x(t + \pi/2) = \cos(t + \pi/2) = \sin(t). \end{aligned}$$

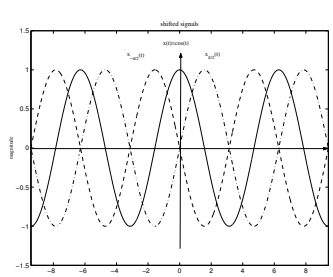


Fig. 16. The signal $x(t) = \cos(t)$.

Example 2: We consider the following signal, which plays important role in digital signal processing,

$$x(t) = e^{-t}, \quad t \geq 0, \quad x(t) = 0, \quad t < 0.$$

The shift by t_0 yields the following amplification of the signal:

$$x_{t_0}(t) = x(t - t_0) = e^{-(t-t_0)} = e^{t_0} e^{-t} = a e^{-t} \tag{6}$$

for $t \geq t_0$, and $x_{t_0}(t) = 0$, otherwise. The amplitude of the time shifted signal at every point of t will increase or decrease for $t \geq t_0$ depending on $a > 1$ or $a < 1$, respectively ($a = e^{t_0}$) (see Fig. 17)

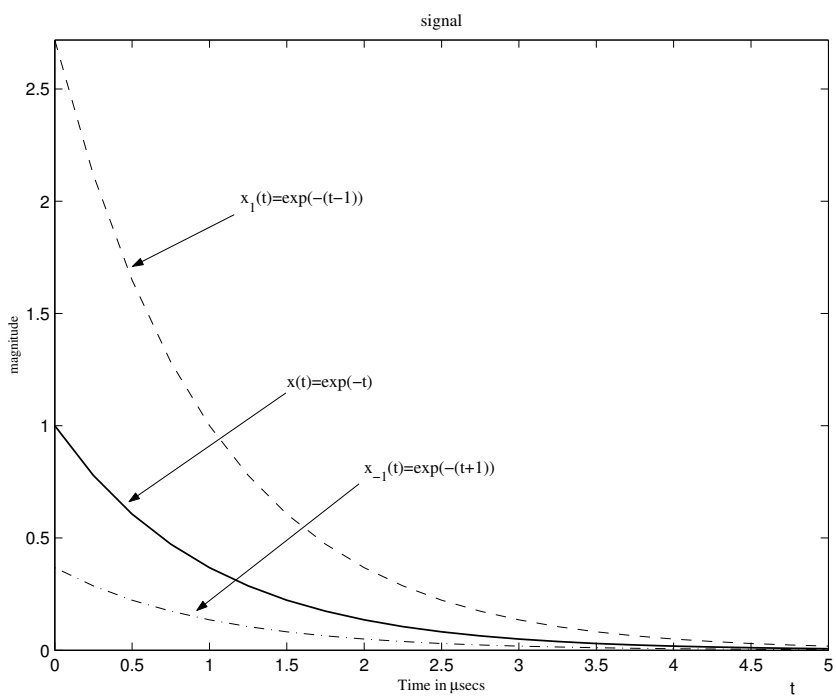


Fig. 17. The shift of the signal $x(t) = e^{-t}$, $t \geq 0$.

Task 1: For signal which is described as

$$x(t) = e^{-t} \cos(t), \quad t \geq 0, \quad x(t) = 0, \quad t < 0,$$

write and plot the time-shifted signals for $t_0 = \pi/2$, $-\pi/2$, π, $\pi/4$, and $-\pi/4$.

3. *Time scaling.* Given a signal $x(t)$ and a positive constant a, a time scaled version of the signal is defined as

$$y(t) = x(at), \quad -\infty < t < +\infty. \tag{7}$$

As an example, Fig. 18 shows the time scaling transformations for $a = 2$ and $a = 1/2$. The signal can be defined on the whole line $R = (-\infty, +\infty)$ which does not change after multiplying by $a \neq 0$, i.e., $aR = R$.

Figure 19 shows the time-scaling of the cosine wave, when the scale factors are 2 and 1/2.

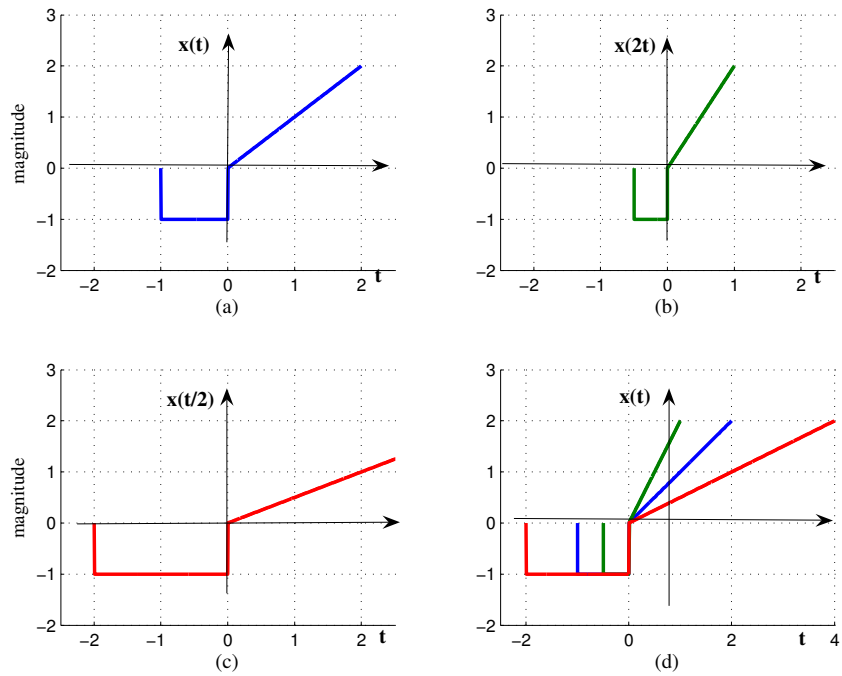


Fig. 18. (a) Signal $x(t)$ and its time-scaling versions for (b) $a = 2$, (c) $a = 1/2$, and (d) all signals together.

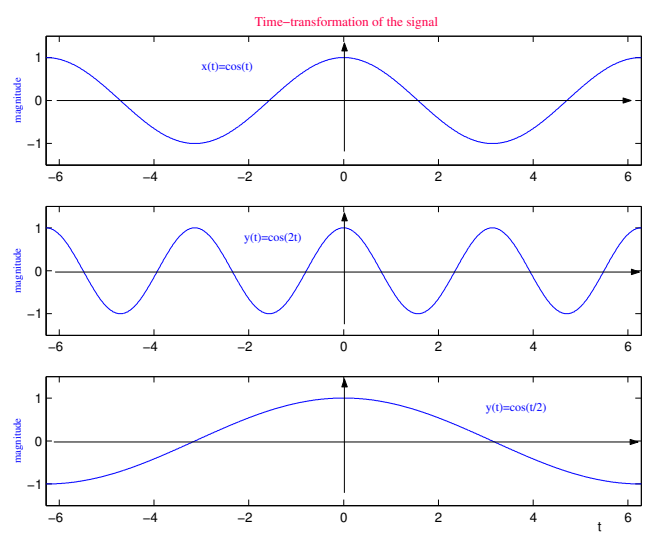


Fig. 19. Cosine signal $x(t) = cos(t)$ and time-scaled versions $y(t) = cos(2t)$ and $y(t) = cos(t/2)$.

3. *Combination of time transformations.*
We consider the general time transformation which is defined as

$$y(t) = x(at - t_0), \quad \infty < t < +\infty, \tag{8}$$

where $a > 0$ is a value of time scaling, t_0 is a value of time shifting. Denoting the new variable

$$t' = at - t_0,$$

we can write that

$$y(t) = x(t'), \quad \text{where} \quad t = (t' + t_0)/a. \tag{9}$$

Example 3: Let $a = -1/2$ and $t_0 = -1$,

$$\begin{aligned} y(t) &= x\left(1 - \frac{1}{2}t\right) = x(t') \\ t &= 2 - 2t' \end{aligned}$$

The construction of $y(t)$ is shown in Fig. 20(c).

We can also plot $y(t)$ by using the following step-by-step transformations:

$$x'(t) = x(-t) \tag{10}$$

$$x'_{1/2}(t) = x'(t/2) \tag{11}$$

$$x'_{1/2}(t-2) = x'([t-2]/2) = x'(t/2 - 1) \tag{12}$$

$$= x(1 - t/2) \tag{13}$$

which is illustrated in Fig. 20.

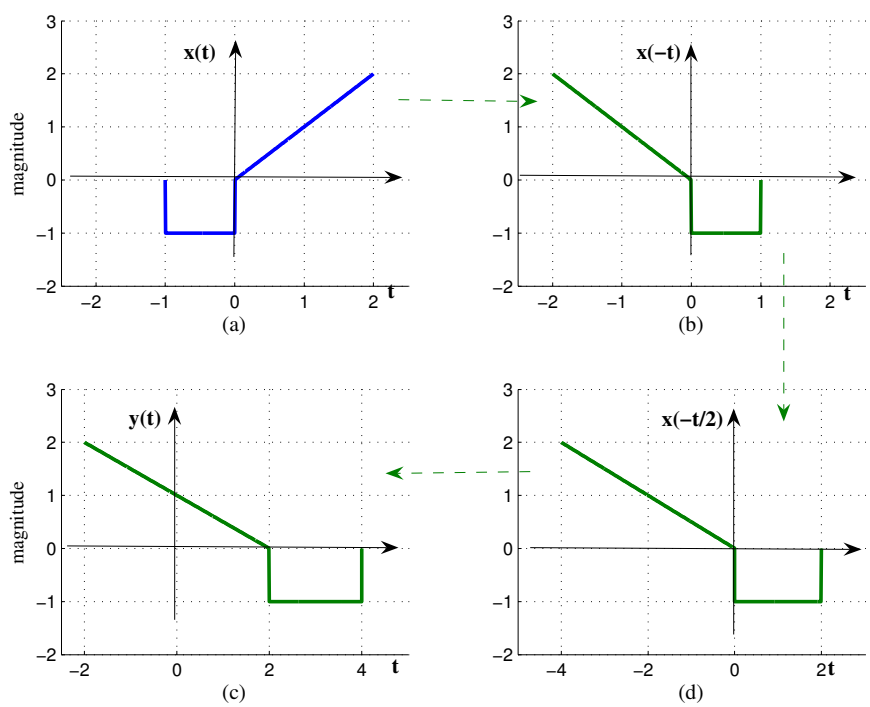


Fig. 20. Construction of the time-transformation $y(t)$ of the signal $x(t)$ in Example 3.

B. Signal-amplitude transformation

5. *Amplitude transformation.* Given a signal $x(t)$, a *amplitude amplification* (AA) is defined as

$$y(t) = Ax(t), \tag{14}$$

where A is a constant. In the general case, the amplitude transformation is defined as

$$y(t) = Ax(t) + B = \pm|A|x(t) + B, \tag{15}$$

where B is another constant to be added (if $B > 0$) or subtracted (if $B < 0$) from the amplified signal in (14). The amplitude transformation signal can also be considered as

$$y(t) = A\left(x(t) + \frac{B}{A}\right) = \pm\left(|A|\left[x(t) + \frac{B}{A}\right]\right), \qquad A \neq 0,$$

i.e., the constant B/A is added to the given signal $x(t)$, and then the signal is amplified by A. In the $A < 0$ case, we observe *amplitude reversal* and amplitude scaling $|A|$, and constant B shifts the amplitude of the amplified signal.

As an example, Figure 21 illustrates the amplitude transformation of the signal $y(t) = 2x(t) + 1$, by using the following two steps of calculations:

$$x(t) \to 2x(t) \to 2x(t) + 1.$$

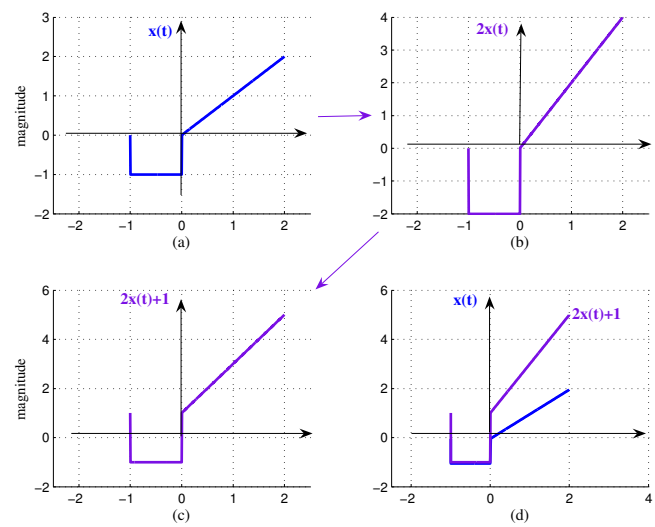


Fig. 21. Signal and its amplitude transforms.

Figure 22 show the same amplitude transformation of the signal, by using other two steps of calculations

$$x(t) \to x(t) + 0.5 \to 2(x(t) + 0.5).$$

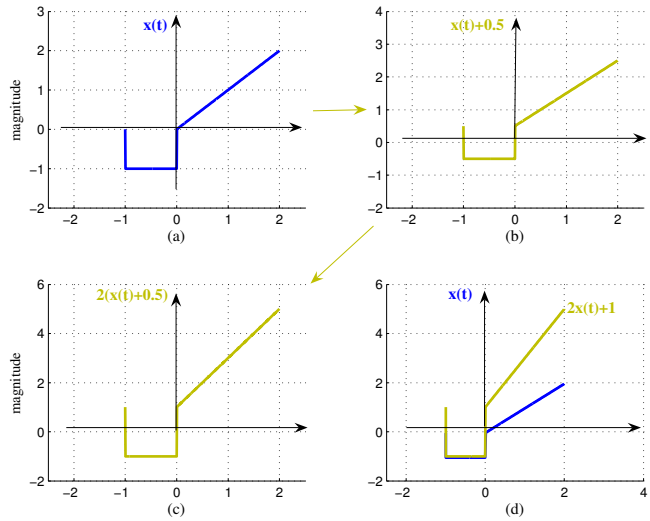


Fig. 22. Signal and its amplitude transforms.

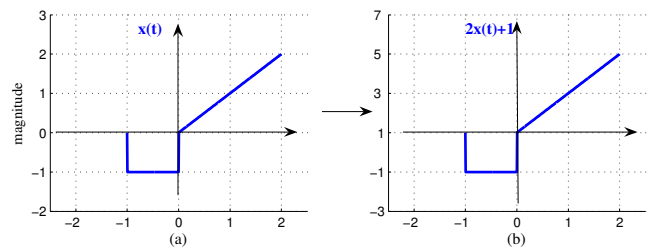


Fig. 23. Signal $x(t)$ and the amplitude transform $2x(t) + 1$.

To draw the graph of the amplitude transform, we can use the original graph and change only numbers along the y-axis. As an example, Figure 23 shows the above amplitude transform of the signal $x(t) \to y(t) = 2x(t) + 1$, by using this method.

Project 1: (Will be given later) Transfer and plot the *transient* signal $x(t) = 2\exp(-\alpha t)\sin(\beta t)$, for $\alpha = 0.02$ and $\beta = 0.25$ which is given on the interval $[0, 255]$ into the window $[0, 255] \times [1, 3]$. This signal is illustrated in Fig. 24 (see also Fig. 6). Use MATLAB and print input $x(t)$ and output $y(t)$ signals.

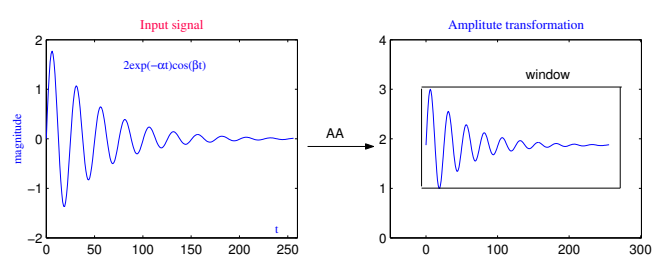


Fig. 24. The time and amplitude transformation of the signal $x(t)$ into the window $[0, 255] \times [1, 3]$.

IV. Signal Characteristics

We now consider the property of symmetry of the signals. Let $x(t)$ be a signal (function) defined on the real line R (or on a symmetric interval $[-1, 1]$, $a > 0$).

(A) $x(t)$ is called *even*, if

$$x(t) = x(-t)$$

for all t; The interval of definition of $x(t)$ should be symmetric. For example, the signals described by the functions $x(t) = |t|$, $x(t) = \cos(t)$, and $x(t) = t\sin(t)$ possess the property of even symmetry, i.e., they have symmetry with respect to the vertical axis $t = 0$, as shown in Fig. 25. For an even signal $x(t)$, the following valid:

$$x(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[x(t) + x(-t)] + \frac{1}{2}[x(t) - x(-t)] \tag{16}$$

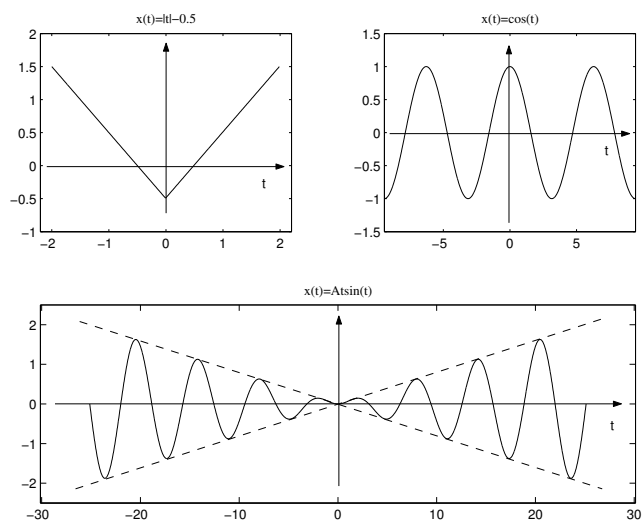


Fig. 25. Even signals.

(B) $x(t)$ is called *odd*, if

$$x(t) = -x(-t)$$

for all t. For example the signals described by the functions $x(t) = t$, $x(t) = \sin(t)$, and $x(t) = t\cos(t)$ possess the property of odd symmetry (see Fig. 26). For an odd signal $x(t)$, the following valid:

$$x(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[x(t) - x(-t)] + \frac{1}{2}[x(t) + x(-t)]. \tag{17}$$

If $x(t)$ is an even (odd) function, then the amplitude transformation

$$y(t) = Ax(t) + B, \quad A \neq 0, \tag{18}$$

is even (not odd, if $B \neq 0$), too. Next, if $x(t)$ is even (odd), then the time reversal signal $y(t) = x(-t)$ is also even (odd). In general, the function $y(t) = x(at)$ has the same evenness as $x(t)$ if $a \neq 0$, but $y(t) = x(at + b)$ does not when $b \neq 0$.

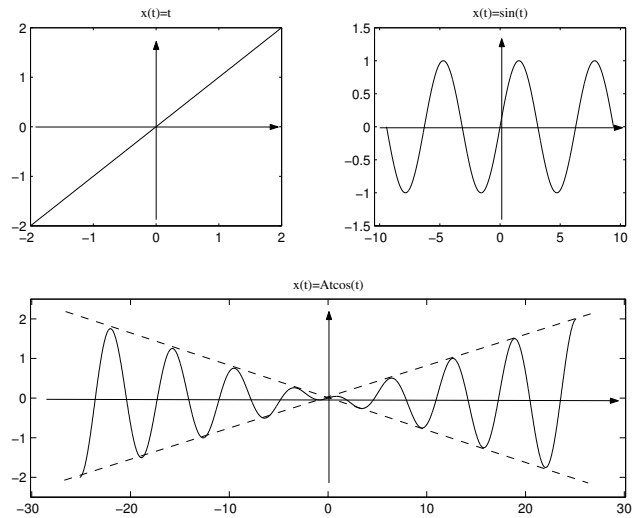


Fig. 26. Odd signals.

The following property is important. An arbitrary signal $x(t)$, $t \in R$, can be represented as the sum of even and odd signals. Indeed, the function $x(t)$ can be written as follows

$$\begin{aligned} x(t) &= \frac{1}{2}\left[x(t)+x(-t)\right]+\frac{1}{2}\left[x(t)-x(-t)\right] \\ &= \frac{1}{2}x^e(t)+\frac{1}{2}x^o(t) \end{aligned}$$

where the even component $x^e(t) = x(t) + x(-t)$ and the odd component $x^o(t) = x(t) - x(-t)$. Indeed

$$\begin{aligned} x^e(-t) &= x(-t)+x(-(-t)) = x(-t)+x(t) = x^e(t), \\ x^o(-t) &= x(-t)-x(-(-t)) = x(-t)-x(t) = -x^o(t). \end{aligned}$$

We denote by $x_e(t)$ and $x_o(t)$ the even and odd parts of the signal, respectively,

$$\begin{aligned} x_e(t) = \frac{1}{2}x^e(-t) &= \frac{1}{2}[x(t)+x(-t)], \\ x_o(t) = \frac{1}{2}x^o(-t) &= \frac{1}{2}[x(t)-x(-t)]. \end{aligned}$$

As an example, Figure 28 shows the even and odd components of the considered above signal.

If $x(t)$ is even, then $x_o(-t) = 0$ and $x(t) = x_e(t)$. If $x(t)$ is odd, then $x_e(-t) = 0$ and $x(t) = x_o(t)$.

We will denote $x_{even}(t) = x(t)$ if $x(t)$ is even and $x_{odd}(t) = x(t)$ if $x(t)$ is odd. Given two signals $x(t)$ and $y(t)$, the following properties hold:

1) $x_{odd} + y_{odd} = (x+y)_{odd}$
2) $x_{even} + y_{even} = (x+y)_{even}$
3) $x_{odd} + y_{even} \neq (x+y)_{even}$
   $x_{odd} + y_{even} \neq (x+y)_{odd}$

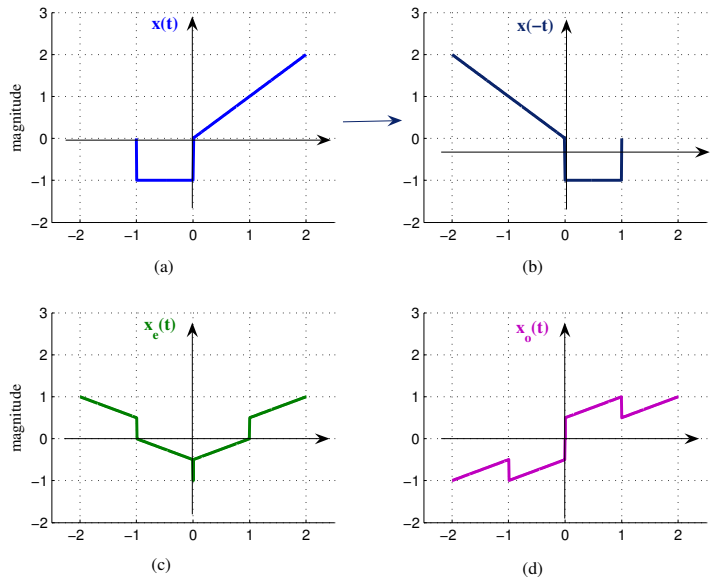


Fig. 27. Decomposition of the signal.

For instance, if $x(t) = \cos(t)$ and $y(t) = \sin(t)$, then $x = x_{even}$ and $y = y_{odd}$, and

$$x(t) + y(t) = \cos(t) + \sin(t) = \sqrt{2}\cos(t + \pi/4)$$

which is neither even nor odd.

4) $x_{even} \times y_{even} = (x \times y)_{even}$
5) $x_{odd} \times y_{odd} = (x \times y)_{even}$
6) $x_{even} \times y_{odd} = (x \times y)_{odd}$

Figure 28 shows the even and odd components of the transient signal $x(t) = 2\exp(-0.02t)\sin(t/4)$.

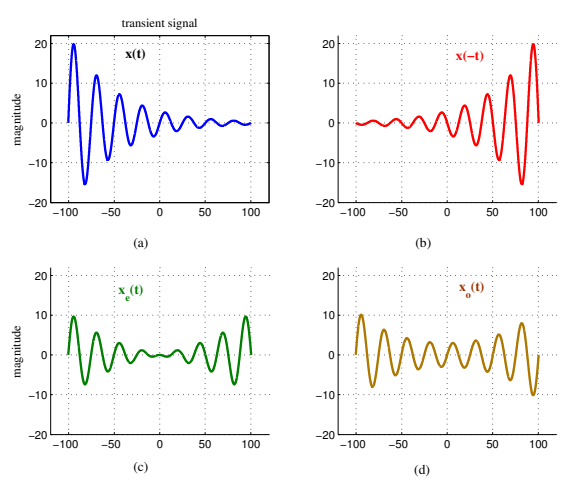


Fig. 28. Decomposition of the transient signal.

V. Periodic signals

A signal $x(t)$ is called *periodic*, if there exist such a constant $T > 0$ that

$$x(t) = x(t+T), \qquad \forall t \in R. \tag{19}$$

T is called a *period* of $x(t)$. By the definition, it is enough to consider the periodic function only on the interval $[0, T)$, because any value t can be written as

$$t = nT + t_0, \quad n \in Z, \quad t_0 \in [0, T),$$

where Z is the set of all integers, $Z = \{0, \pm 1, \pm 2, ...\}$, and $x(t) = x(t_0)$. Indeed, the following is valid:

$$\begin{aligned} x(t) &= x(nT + t_0) = x((n-1)T + t_0 + T) \\ &= x((n-1)T + t_0) = ... = x(T + t_0) = x(t_0). \end{aligned}$$

$-T$ is also period of the signal, since $x(t-T) = x((t-T)+T) = x(t)$. Any integer multiple of the period, $\pm kT$, where k is an integer, is a period of the signal,

$$x(t \pm T) = x(t \pm 2T) = \cdots = x(t \pm kT) = x(t), \quad \forall t \in R.$$

The smallest period $T > 0$ of the signal $x(t)$ is called a *fundamental period* of the signal, $x(t) = x(t+T)$. [1]

As an example, Figure 29 shows the periodic signal with the fundamental period $T = 6$ in the interval $[-18, 18]$, i.e., six periods of the signal.

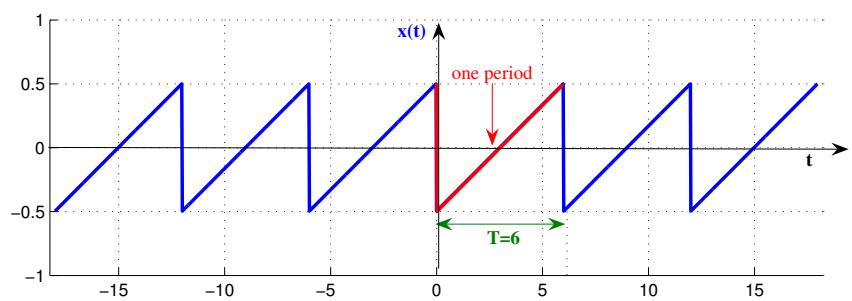


Fig. 29. Periodic signal and its one period.

We also use the concept of a *fundamental* frequency in hertz, which shows how many fundamental periods are placed in the interval of 1 second

$$f_0 = \frac{1}{T_0} Hz \tag{20}$$

and in radians the frequency is defined as

$$\omega = 2\pi f_0 = \frac{2\pi}{T_0} \frac{rad}{\text{sec}}. \tag{21}$$

[1] Another definition: The fundamental period is the interval $[0, T)$.

For the periodic signal of Fig. 29, one can see that there is only 1/6 part of the signal in the interval of 1 second. Thus the frequency equals

$$f_0 = \frac{1}{6} Hz, \quad \text{or} \quad \omega = \frac{\pi}{3} \frac{rad}{\text{sec}}.$$

Example 1: Signals $x(t) = \cos(t)$ has the fundamental period equal $T = 2\pi$, and the signal $x_1(t) = \cos(4t)$ has the period $T_1 = \pi/2$, i.e., $T_1 = T/4$. The time-scaling transformations $t \to 4t$ transfers one period $[0, 2\pi)$ of $x(t)$ into the interval $[0, 2\pi)/4 = [0, \pi/2)$, as shown in Fig. 30.

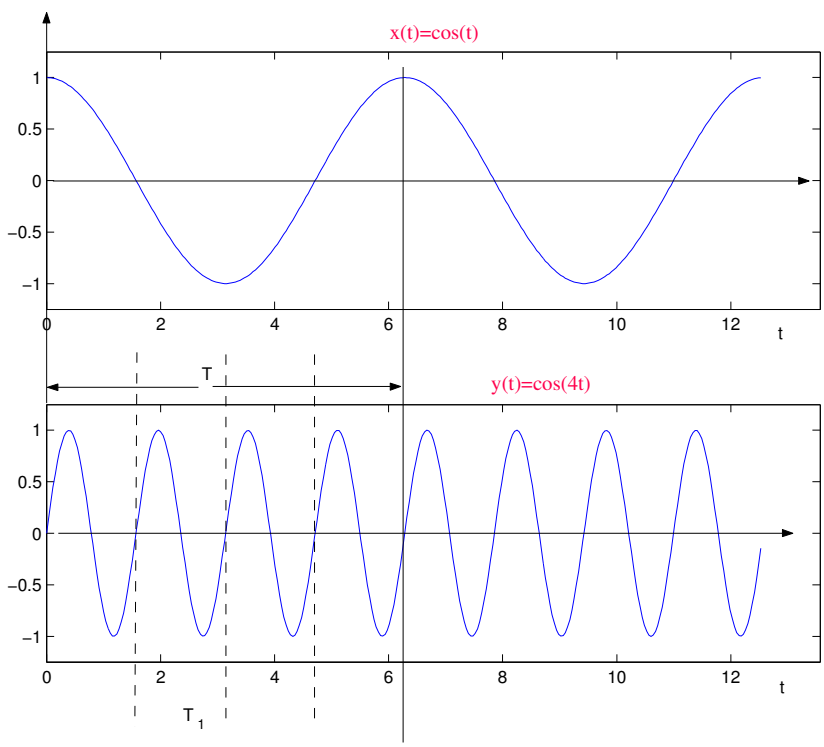


Fig. 30. Two periodic cosine functions (shown only in the interval $[0, 12]$).

Example 2: We now consider the periodic signal called the *sawtooth wave* (shown in Figure 31), which is useful in sweeping a beam of electrons across of face of CRT (cathode ray tube),

$$x(t) = t_0 = t \bmod 1, \tag{22}$$

if $t = n + t_0$, where $t_0 \in [0, 1)$ and $n \in Z$.

It is clear, that if

$$x(t+T) = x(t), \quad y(t+T) = y(t)$$

then

$$\begin{aligned} (x+y)(t+T) &= x(t+T) + y(t+T) \\ &= x(t) + y(t) = (x+y)(t) \end{aligned}$$

i.e., the sum of two periodic signals with the period T is also periodic with the period T.

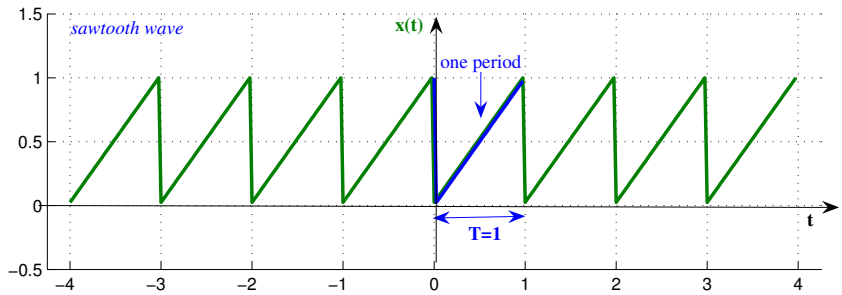


Fig. 31. Eight periods of the sawtooth wave.

In the general case, when periods are different,

$$x_1(t+T_1) = x_1(t), \quad x_2(t+T_2) = x_2(t), \quad T_2 \neq T_1,$$

the sum of two periodic functions (signals) $x(t) = x_1(t) + x_2(t)$ not necessary to be periodic. This function is periodic if only $x_1(t)$ and $x_2(t)$ have a common period. If these two signals have a common period at a point T on the line, then

$$T = n_1 T_1 = n_2 T_2 \tag{23}$$

for some integers n_1 and n_2. In this case T is period of $x(t)$. If T is the first positive such point, then T is the fundamental period.

Example 3: Signals $x_1(t) = \cos(t)$ has the period $T_1 = 2\pi$, and the signal $x_2(t) = \cos(2t)$ has the period $T_2 = \pi$. We have $1T_1 = 2T_2 = 2\pi$, therefore the signal $x(t) = \cos(t) + \cos(2t)$ has the period $T = 2\pi$, as shown in Fig. 32.

Example 4: Consider signals $x_1(t) = 2\cos(2t)$ with period $T_1 = \pi$, and the signal $x_2(t) = 5\sin(3t)$ has period $T_2 = 2\pi/3$. The signal $x(t) = 2\cos(2t) + 5\sin(3t)$ is periodic and its fundamental period $T = 2\pi$, as shown in Fig. 33. Indeed, we have the following

$$2T_1 = 3T_2 = 2\pi \to T = 2\pi,$$

i.e., we solve (23) with $n_1 = 2$ and $n_2 = 3$.

Example 5: We now consider power-supply periodic signals which convert the sinusoidal voltage (see Fig. 34 in part a) into the constant voltage (in c). For a given constant $T_0 > 0$, we consider the following non negative signal (called a *full-wave rectified* signal (shown in b)

$$x(t) = \left|\sin\left(\frac{\pi t}{T_0}\right)\right|, \quad t \in R. \tag{24}$$

The half-wave rectified signal is defined as $y(t) = x(2t)u(t)$, where $u(t)$ is the binary function described the unit signals (impulse) with period $T_0/2$,

$$u(t) = \begin{cases} 1, & \text{if } t = 2n(T_0/2) + t_0, \quad t_0 \in [0, T_0/2); \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

The construction of the full-wave rectified signal is illustrated in Figs. 34 and 35.

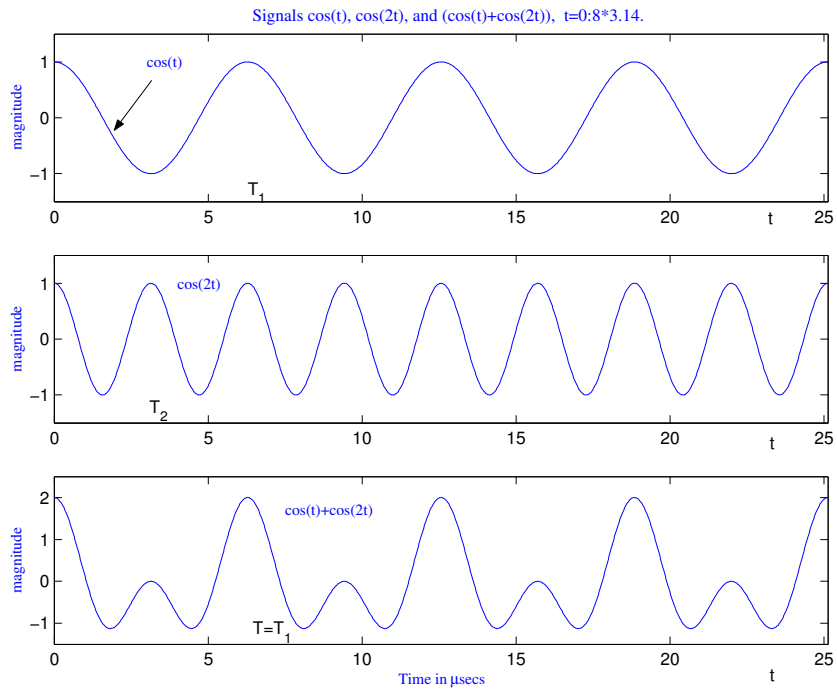


Fig. 32. The sum of two periodic functions.

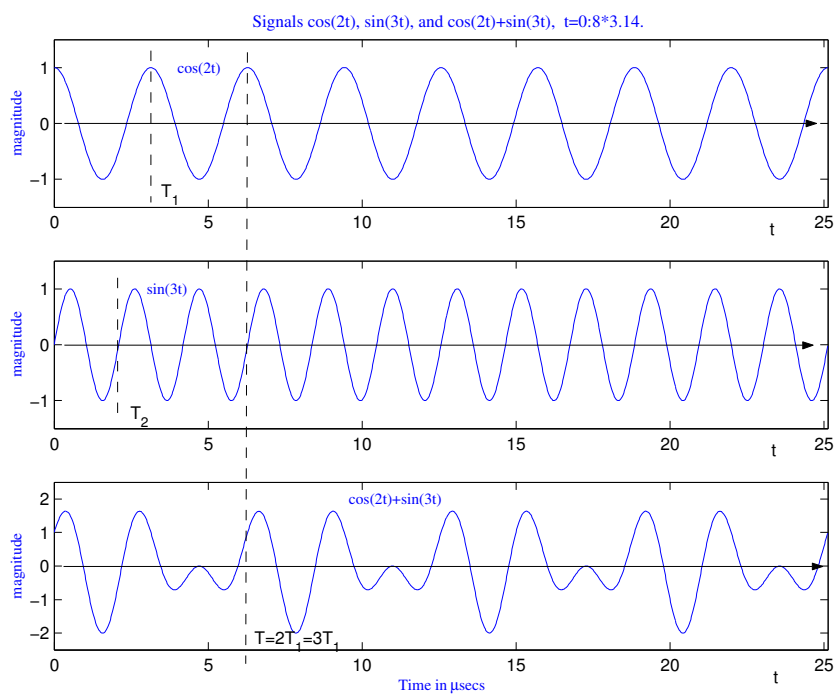


Fig. 33. Two cosine functions with the common period at point $T = 2T_1 = 3T_2 = 2\pi$.

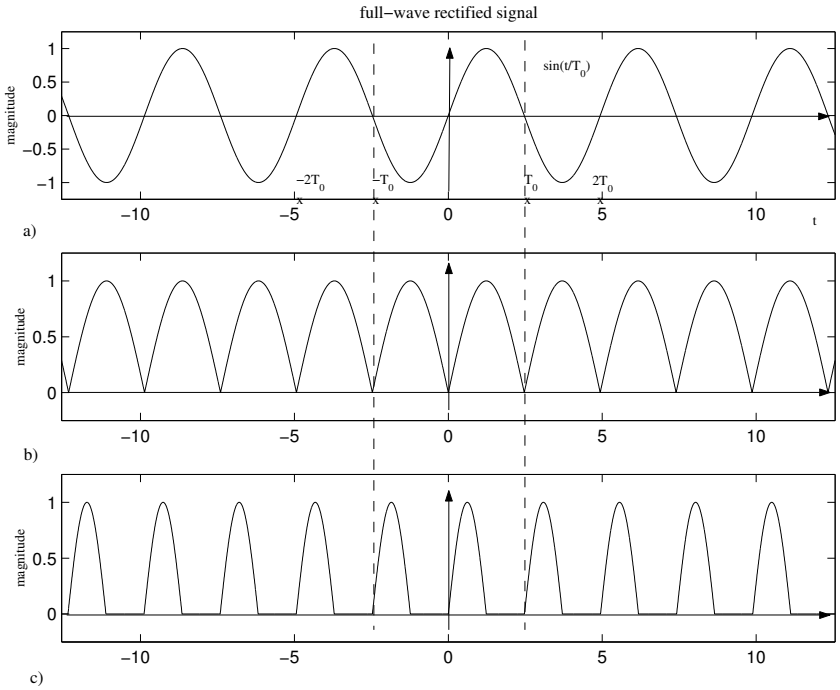


Fig. 34. Power-supply periodic signal construction (a)-(c).

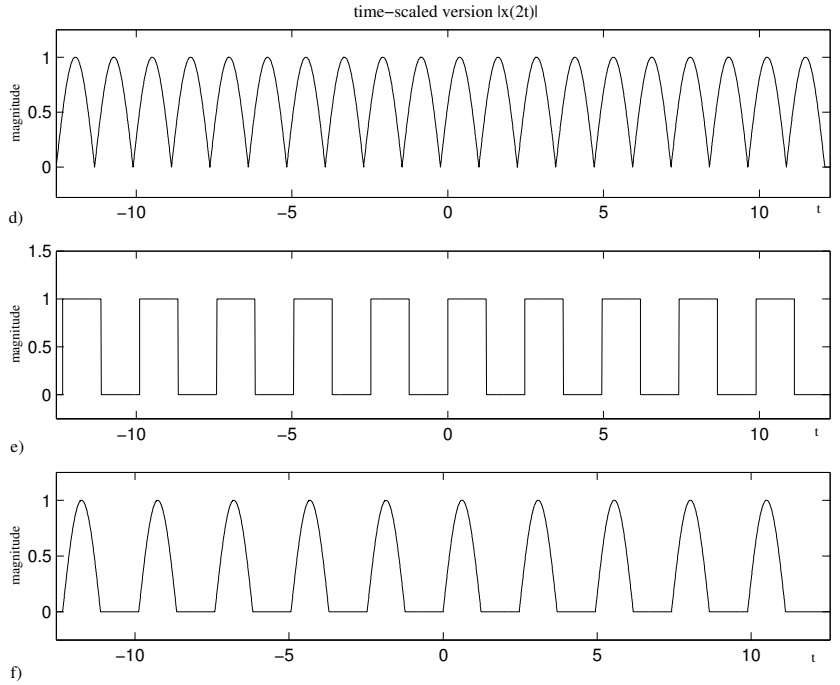


Fig. 35. Power-supply periodic signal construction (d)-(f).

A. Discrete-time periodic signals

We consider the process of sampling of a periodic signal

$$x(t) = x(t + nT), \quad \forall t \in R,$$

where T is the period of $x(t)$.

Let T_0 is the sampling interval, i.e., in every T_0 seconds, the value $x_n = x(nT_0)$ will be computed. In order to reach into one of the periods, rT, after n intervals of time T_0, the following equation should have place

$$\begin{aligned} nT_0 &= rT = r\frac{2\pi}{\omega_0}, \quad \exists r \in Z, \\ n\omega_0 T_0 &= 2\pi r. \end{aligned}$$

If $T_0 = 1$ then $n\omega_0 = 2\pi r$, and $n = 2\pi/\omega_0 = Tr$.

If T_0 is an integer, then one can take $r = 1$ and $N = T$.

In the discrete case, the sum of two periodic (discrete-time) signals is a periodic function (or, signal)

$$\begin{aligned} x_n &= x_{n+N}, \quad N \text{ is the period} \\ y_n &= y_{n+M}, \quad M \text{ is the period} \end{aligned}$$

There exist an integer K such that

$$\begin{aligned} x_n &= x_{n+K}, \quad \text{for } \forall n \in Z \\ y_n &= y_{n+K}, \quad \text{for } \forall n \in Z \end{aligned}$$

For instance, we can take $K = NM$.

Indeed, for any integer n,

$$\begin{aligned} x_{n+K} &= x_{n+MN} = x_{n+(M-1)N+N} = x_{n+(M-1)N} \\ &= x_{n+(M-2)N+N} = x_{n+(M-2)N} = \dots \\ &= x_{n+N} = x_n \end{aligned}$$

$$\begin{aligned} y_{n+K} &= y_{n+NM} = y_{n+(N-1)M+M} = y_{n+(N-1)M} \\ &= y_{n+(N-2)M+M} = y_{n+(N-2)M} = \dots \\ &= y_{n+M} = y_n \end{aligned}$$

So, K is a period of the sum of discrete-time signals (but may be not the fundamental period).

B. Time constant

To define the concept of time constant, we consider the solution in the general form $x(t) = Ce^{-at}$, when $a > 0$, and draw the tangent to the curve of $x(t)$ at point $t = 0$ (see Fig. 37). We have

$$x'(0) = -Ca(e^{-at})_{|t=0} = -Ca = -\frac{C}{\tau} \quad \rightarrow \quad \tau = \frac{1}{a}$$

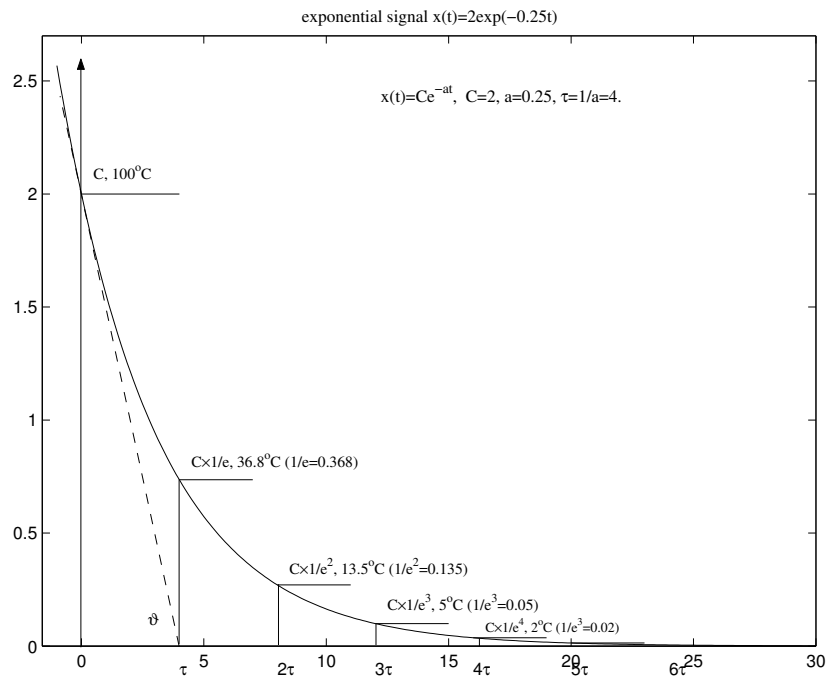


Fig. 36. Exponential and time constant.

Taking value $x(t)$ at time t, the exponential decays to less than 36.8% of its amplitude in 1τ unit of time, $t+\tau$. That is, $x(t+\tau) \approx 0.368x(t), \ \ldots, \ x(t+5\tau) = 0.00067x(t)$.

In general, we can write that the following take place for any integer $n > 1$:

$$x(t+n\tau) \approx 0.368x(t+(n-1)\tau) \approx 0.368 \times [0.368x(t+(n-2)\tau)] \approx \ \cdots \ . \tag{26}$$

VI. Sinusoidal signals in engineering

In many cases, it is useful to transfer our calculations from the real space to complex space, analyze and solve problems by using methods of the complex analysis (arithmetic), and then transfer back the solution to the real space.

Below is a brief introduction to the complex arithmetic which we should know well in order to understand better the main concepts of Fourier transforms and the frequency analysis of functions and signals by the Fourier transforms.

A. Complex Arithmetic

We first go back to XVI century, when the formal and not real solution of the simple quadratic equation

$$z^2 + 1 = 0, \qquad \text{or} \qquad z^2 = -1, \tag{27}$$

was proposed by two Italian mathematicians Rafael Bombelli and Gerolamo Cardano (XVI century) and denoted by $z = \sqrt{-1}$. Another solution of this equation is $z = -\sqrt{-1}$.

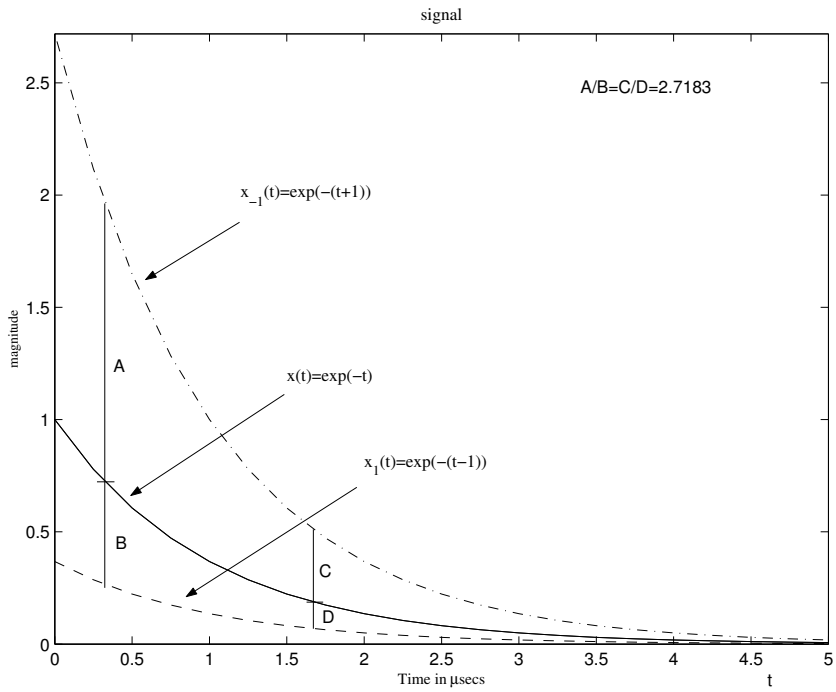


Fig. 37. Exponential and time constant.

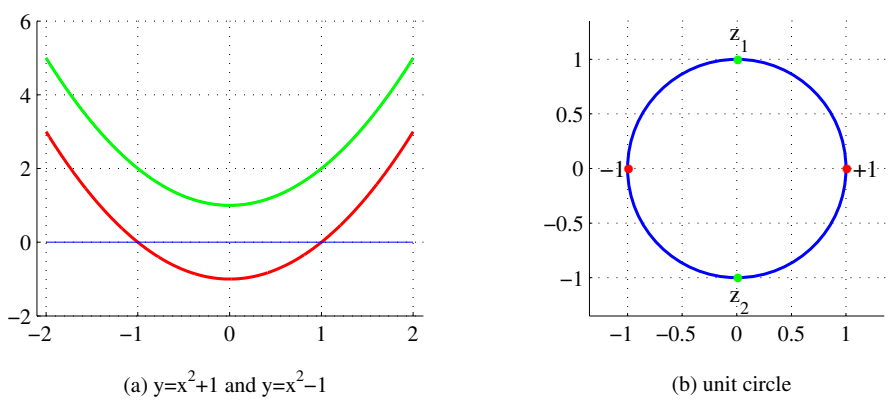


Fig. 38. (a) Parabola and (b) the unit circle with four unit points on it.

Figure 38 shows the graph of the parabola $y = x^2 + 1$ in part a, which does not intersect the horizontal line, i.e., there is no solution of equation (27) in the real arithmetic. Solutions of this equation should be found in another arithmetic which will be described in a moment; in fact, these two solutions are on the unit circle shown in part b. There are four unit points on this circle. Two points are on the horizontal line, ± 1, and they are the solutions of equation $z^2 - 1 = 0$, which can be written as $x^2 - 1$ with real numbers x. The graph of the parabola $y = x^2 - 1$ is also shown in part a. Two other points of the circle, which are on the vertical line, are the solutions of equation $z^2 + 1 = 0$. These points $z_1 = \sqrt{-1}$ and $z_2 = -\sqrt{-1}$.

Given real number y, the solutions of the equation

$$z^2 + y^2 = 0, \quad \text{or} \quad \left(\frac{z}{y}\right)^2 + 1 = 0, \tag{28}$$

can be written as $z/y = \pm\sqrt{-1}$, or $z = \pm y\sqrt{-1}$, or $z = \pm(\sqrt{-1})y$. If for a given real number x we consider the quadratic equation

$$(z - x)^2 + y^2 = 0, \tag{29}$$

then, one can see that $z - x = \pm y\sqrt{-1}$ and the solutions of this equation are the numbers $z = x + y\sqrt{-1} = x + (\sqrt{-1})y$ and $z = x - y\sqrt{-1} = x - (\sqrt{-1})y$.

In XVIII century, Euler denoted this imaginary number or symbol $\sqrt{-1}$ by i, i.e., $i = \sqrt{-1}$ and $i^2 = -1$. This symbol represents an imaginary unit, and in the engineering community it is also denoted by j, since the letter "i" is used for the electrical current. The number $z = x + iy = x + yi$ is called *a complex number*. Numbers x and y are real, x is called the real part of z and y is the imaginary part of z. The concept of the complex number generalizes the real numbers which can be considered as the complex numbers with zero imaginary part, i.e., when $y = 0$. The arithmetical operations are also generalized in the complex arithmetic and we consider the main operations over complex numbers.

Given complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the following properties for operations of addition and multiplication are valid:

1. $z_1 + z_2 = [x_1 + iy_1] + [x_2 + iy_2] = (x_1 + x_2) + i(y_1 + y_2)$,
2. $kz_1 = k[x_1 + iy_1] = (kx_1) + i(ky_1)$

for any real number k. One can note, that $z_1 + z_2 = z_2 + z_1$. The operation of subtraction is defined as $z_1 - z_2 = z_1 + (-1)z_2$,

3. $z_1 - z_2 = [x_1 + iy_1] - [x_2 + iy_2] = (x_1 - x_2) + i(y_1 - y_2)$.

The most important operation of complex numbers is the multiplication, $z = z_1 z_2$ which is calculated directly as

$$z_1 z_2 = [x_1 + iy_1][x_2 + iy_2] = x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2.$$

Considering the definition, $i^2 = -1$, we obtain the following:

4. $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$.

Thus, the multiplication of two complex numbers z_1 and z_2 is the complex number $z = x + iy$ with real and imaginary parts defined by $x = x_1x_2 - y_1y_2$ and $y = x_1y_2 + y_1x_2$, respectively. When the numbers z_1 and z_2 are real, i.e., $y_1 = 0$ and $y_2 = 0$, this operation is reduced to the multiplication of real numbers, $z_1z_2 = x_1x_2$. The set of complex numbers is denoted by C.

Example 6: If $z_1 = 1 + 2i$ and $z_2 = 2 - 3i$, then the multiplication

$$z_1z_2 = [2 - 2(-3)] + i[-3 + 2(2)] = 8 + i.$$

The operation of multiplication together with operations of addition and subtraction is commutative, i.e.,

5. $z_2z_1 = (x_2x_1 - y_2y_1) + i(x_2y_1 + y_2x_1) = z_1z_2$.

It is not difficult to see that for any real numbers k_1 and k_2 and complex numbers z_1, z_2, and z_3, the following holds:

$$5a. \quad (k_1 z_1)(k_2 z_2) = (k_1 k_2)(z_1 z_2),$$
$$5b. \quad z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

For a complex number $z_1 = x_1 + iy_1$, the number $z_2 = x_1 - iy_1$ is called *the complex conjugate* and denoted by $\bar{z}_1$. It is clear that, $\bar{\bar{z}}_1 = z_1$, for any complex number, and $\bar{z}_1 = z_1$, if only the number z_1 is real. The operation of conjugation $z \to \bar{z}$ is a linear operation, namely

$$6. \quad \overline{z_1 + kz_2} = \bar{z}_1 + k\bar{z}_2,$$

for any real number k and complex numbers z_1 and z_2.

The module $|z_1|$ of the complex number z_1 is defined as the multiplication $z_1\bar{z}_1$, which according to multiplication can be written as

$$6a. \quad z_1\bar{z}_1 = (x_1x_1 + y_1y_1) + i(-x_1y_1 + y_1x_1) = x_1^2 + y_1^2$$

and denoted as $|z_1|^2$. Therefore, $|z_1| = \sqrt{x_1^2 + y_1^2}$ and it is positive if $z_1 \neq 0$, and $|z_1| \geq |x_1|$ and $|z_1| \geq |y_1|$.

In the general case, the following holds for the complex conjugate of the multiplication:

$$6b. \quad \overline{z_1 z_2} = (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2) = \bar{z}_1\bar{z}_2.$$

The following equalities hold for the multiplication:

$$|z_1z_2|^2 = (z_1z_2)\overline{(z_1z_2)} = (z_1z_2)(\bar{z}_1\bar{z}_2) = (z_1\bar{z}_1)(z_2\bar{z}_2) = |z_1|^2|z_2|^2.$$

Therefore, the length of the product of two complex numbers equals the product of their lengths

$$6c. \quad |z_1z_2| = |z_1||z_2|.$$

Example 7: For the complex number $z_1 = 3 + 4i$, the length of the number is

$$|z_1| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5.$$

The complex conjugate number $\bar{z}_1 = 3 - 4i$ has the same length,

$$|\bar{z}_1| = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The property of triangle inequality holds for the complex numbers

$$6d. \quad |z_1 + z_2| \leq |z_1| + |z_2|.$$

Th equality holds for the cases one of the complex numbers is zero, or the numbers are real and have the same sign.

When z_1 is an imaginary number, i.e. $x_1 = 0$, the square of the number is $z_1^2 = (iy_1)^2 = -y_1^2$, i.e.,

$$6e. \quad z_1^2 = -|z_1|^2.$$

Example 8: For the complex number $z_1 = 4i$, the length of the number is

$$|z_1| = \sqrt{4^2} = 4, \quad |z_1|^2 = 16, \quad \text{and} \quad z_1^2 = (4i)^2 = 16i^2 = -16.$$

From equation $z_1\bar{z}_1 = |z_1|^2$, the inverse number $1/z_1$ is defined as

$$7. \quad (z_1)^{-1} = \frac{1}{z_1} = \frac{1}{|z_1|^2}\bar{z}_1, \quad \text{when} \quad z_1 \neq 0.$$

Indeed, one can verify that $(z_1)^{-1}(z_1) = (z_1)(z_1)^{-1} = 1$. Therefore, the operation of division of the complex numbers z_2 and z_1, if only $z_1 \neq 0$, is defined as

$$8. \quad \frac{z_2}{z_1} = z_2(z_1)^{-1} = z_2\frac{1}{z_1} = \frac{1}{|z_1|^2}z_2\bar{z}_1 = \frac{z_2\bar{z}_1}{|z_1|^2}.$$

Example 9: For the complex numbers $z_1 = 3 - 4i$, and $z_2 = 2 + i$, we have the following:

$$\frac{1}{z_1} = \frac{1}{3-4i} = \frac{1}{(3^2+4^2)}\overline{(3-4i)} = \frac{1}{25}(3+4i) = \frac{3}{25} + i\frac{4}{25}$$

and

$$\frac{z_2}{z_1} = \frac{2+i}{3-4i} = \frac{1}{25}(2+i)(3+4i) = \frac{1}{25}(2+11i) = \frac{2}{25} + i\frac{11}{25}.$$

In this complex arithmetic, we can solve equations (27)-(29). For that, we first consider the complex numbers z on the imaginary line, i.e., $z = iy$. The solution of equation $z^2 + 1 = 0$, which for such complex numbers is $-y^2 + 1 = 0$, are $z = i(+1) = i$ and $z = i(-1) = -i$. Figure 39 shows the graph of the parabola $z^2 + 1 = (iy)^2 + 1$ in part a, when iy runs the interval of imaginary numbers $[-2i, 2i]$. One can see this parabola intersects the horizontal at points $+i$ and $-i$. Another

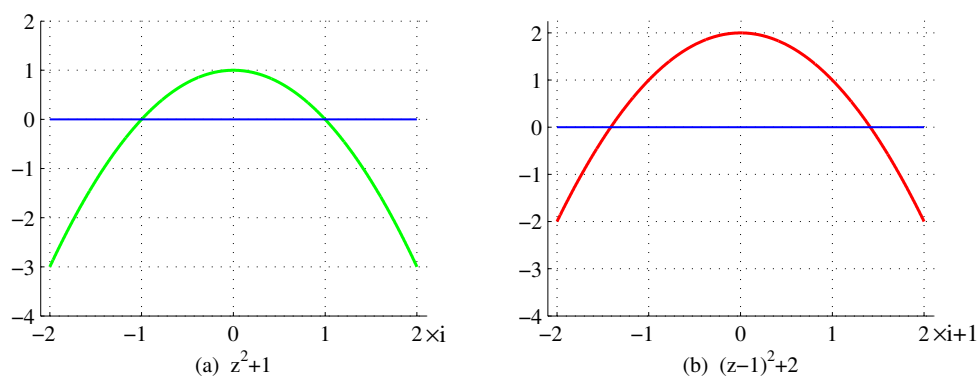


Fig. 39. Graphs of (a) the parabola $z^2 + 1$ and (b) the parabola $(z-1)^2 + 2$.

parabola $(z-1)^2 + 2$ is also shown in part b. It should be noted, that the graph of this parabola is calculated for the complex numbers $z = 1 + iy$ when the imaginary components y runs the interval $[-2, 2]$. Therefore, the plot is shown versus these complex numbers $z = 1 + iy$. This parabola intersects the horizontal line in two points, $z_1 = 1 + i\sqrt{2}$ and $z_2 = 1 - i\sqrt{2}$.

A.1 Geometry of complex numbers

Every complex number $z = x + iy$ can uniquely be presented as the point (x, y) on the real plane R^2. In fact, it is another form of writing the complex number. We denote this point as $P = P(z)$

which has Cartesian coordinates (x, y). The writing, $P = x + jy$ relates to the complex arithmetic. The distance between the original and P is $d(P) = |z| = \sqrt{x^2 + y^2}$. So, the point P lies on the circle with radius $r = d(P)$.

In particular $r = 1$ case, when point P lies on the unit circle, we can write P as

$$P = P(\theta) = (x, y) = (\cos\theta, \sin\theta) = \cos\theta + i\sin\theta$$

where the imaginary unite $i^2 = -1$. Similarly, the point $\bar{P} = P(-\theta)$ which represent the complex conjugate number $\bar{z}$ can be written as

$$\bar{P} = P(-\theta) = (x, -y) = (\cos\theta, -\sin\theta) = \cos\theta - i\sin\theta.$$

Both points P and $\bar{P}$ are on the same circle of radius $r = d(P) = d(\bar{P})$. Therefore,

$$\begin{aligned} P(\theta) + P(-\theta) &= 2\cos\theta, \\ P(\theta) - P(-\theta) &= i2\sin\theta. \end{aligned}$$

The function $P(\theta)$ is denoted by $e^{i\theta}$ in honor of Euler, who used this function and founded the above relations for real and imaginary parts of $e^{i\theta}$,

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta, \tag{30}$$
$$e^{i\theta} - e^{-i\theta} = i2\sin\theta. \tag{31}$$

The point on the unit circle is completely described by the angle θ. In the general case, when point P lies on the circle of radius r, the point can be represented as (r, θ). It is the so-called *polar*

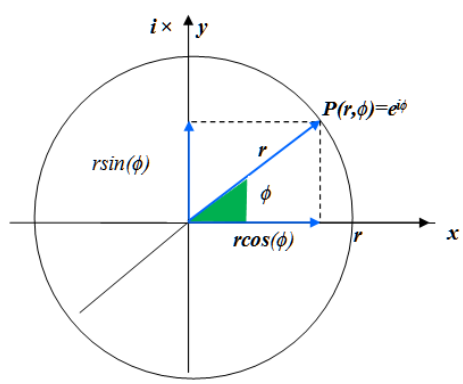


Fig. 40. Point $P(r, \theta)$ on the complex plane C.

form of representation of P (as shown in Fig. 40) instead of $P = (x, y) = (r\cos\theta, r\sin\theta)$,

$$P = P(r, \theta) = (r, \theta).$$

Here the radius and angle are calculated as

$$r = \sqrt{x^2 + y^2}, \quad \text{and} \quad \theta = \arctan\frac{y}{x} = \tan^{-1}\frac{y}{x},$$

and $\theta = \arccos(y) = \pm\pi/2$, if $x = 0$. This angle θ is also called the argument of z and denoted by $\theta = \arg(z)$. If the point P lies in the left semi-plane, the calculation of the angle θ should be corrected by adding the angle $\pm\pi$ depending on the sign of the imaginary part y.

Example 10: The complex numbers $z = 3 + 4i$ as point $P(z) = (3, 4)$ is described in the polar form as $(r, \theta) = (5, 0.9273)$. Indeed,

$$r = \sqrt{3^2 + 4^2} = 5 \quad \text{and} \quad \theta = \arctan\frac{4}{3} = 0.9273 \text{ (in radians)},$$

or

$$\theta = \frac{180}{\pi} \times \arctan\frac{4}{3} = 53.1301^\circ \text{ (in degrees)}.$$

The polar form for the complex conjugate $\bar{z} = 3 - 4i$ is $P(\bar{z}) = (5, -0.9273)$,

$$r = \sqrt{3^2 + (-4)^2} = 5 \quad \text{and} \quad \theta = \arctan\frac{-4}{3} = -0.9273 \text{ (in radians)},$$

or $\theta = -53.1301^\circ$ in degrees.

B. Simple Differential Equation

The resistance-inductance (R-L) circuit shown in Fig. 41 is described by the following linear differential equation

$$Li'(t) + Ri(t) = 0 \tag{32}$$

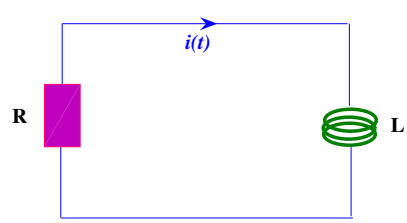


Fig. 41. Symbol of the (R-L) circuit.

Let us consider the general differential equation

$$x'(t) = ax(t) \tag{33}$$

The solution of this equation is founded in the exponential form

$$x(t) = Ce^{\lambda t} \tag{34}$$

where the constants C and λ are defined from (32), because the derivative of the exponent is also exponent $(e^t)' = e^t$. Therefore,

$$x'(t) = \left(Ce^{\lambda t}\right)' = C\lambda e^{\lambda t} = \lambda(Ce^{\lambda t}) = \lambda x(t)$$

which yields $\lambda = a$ and the solution is

$$x(t) = Ce^{at}, \quad C = x(0). \tag{35}$$

In the R-L circuit case, when $a = -R/L$, the solution is

$$i(t) = Ce^{R/Lt}, \quad C = i(0). \tag{36}$$

. . . . . .

$$g(x) = \begin{cases} a^n; & \text{if } n \geq 0 \\ 0; & \text{if } n < 0 \end{cases}$$

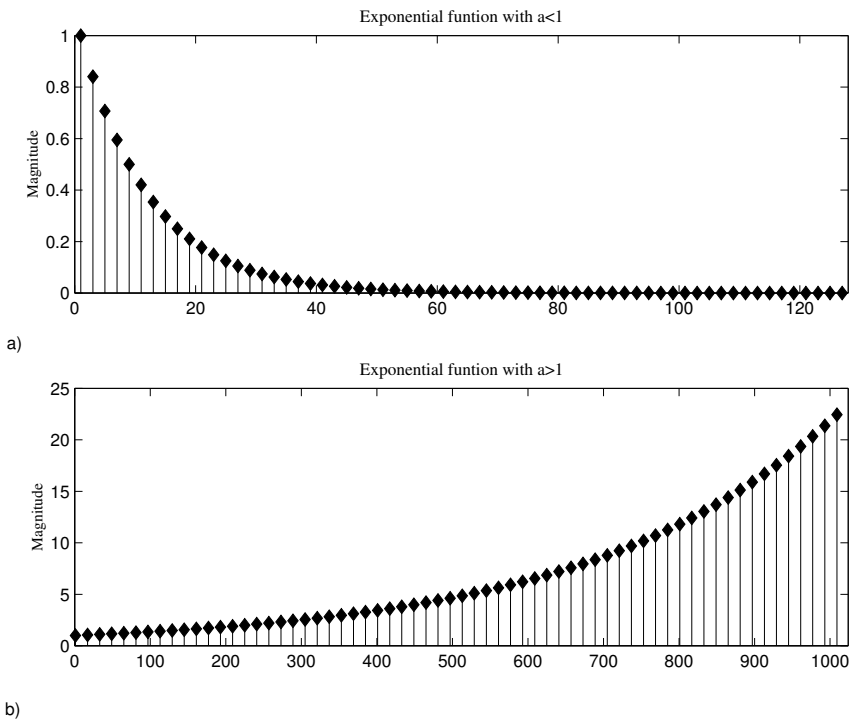


Fig. 42. The exponential sequences $g(n)$, for (a) $a = 0.5$ and (b) $a = 1.025$.

. . . . . .

Remark 1: The polar form is used (only) for presentation of points on the plane

$$(x, y) = (r\cos\vartheta, r\sin\vartheta) \to (r, \vartheta)$$

$$x(t) = Ce^{\alpha t} = Ce^{j\omega t}, \qquad \alpha = j\omega$$
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t), \qquad j^2 = -1,$$

only if ω is real.

B.1 *Complex exponential signal with complex amplitude*

$$x(t) = Ce^{j\omega_0 t}, \quad C = e^{j\phi}A$$

where ϕ, ω_0, and A are real constants. This complex signal can be written as

$$x(t) = Ae^{j\phi}e^{j\omega_0 t} = Ae^{j(\omega_0 t+\phi)}$$

$$x(t) = (A\cos(\omega_0 t + \phi), A\sin(\omega_0 t + \phi))$$

B.2 *Complex exponential signal with exponential amplitude*

We consider the complex signal with amplitude to be an exponential function of time

$$x(t) = Ce^{j\omega_0 t}, \quad C = C(t) = A(t)e^{j\phi}, \quad A(t) = A_0 e^{\sigma_0 t}.$$

The signal $x(t)$ has the form $x(t) = A_0 e^{\sigma_0 t}e^{j(\omega_0 t+\phi)}$ and can be written as (see Fig. 43)

$$x(t) = \Big(Ae^{\sigma_0 t}\cos(\omega_0 t + \phi), Ae^{\sigma_0 t}\sin(\omega_0 t + \phi)\Big)$$

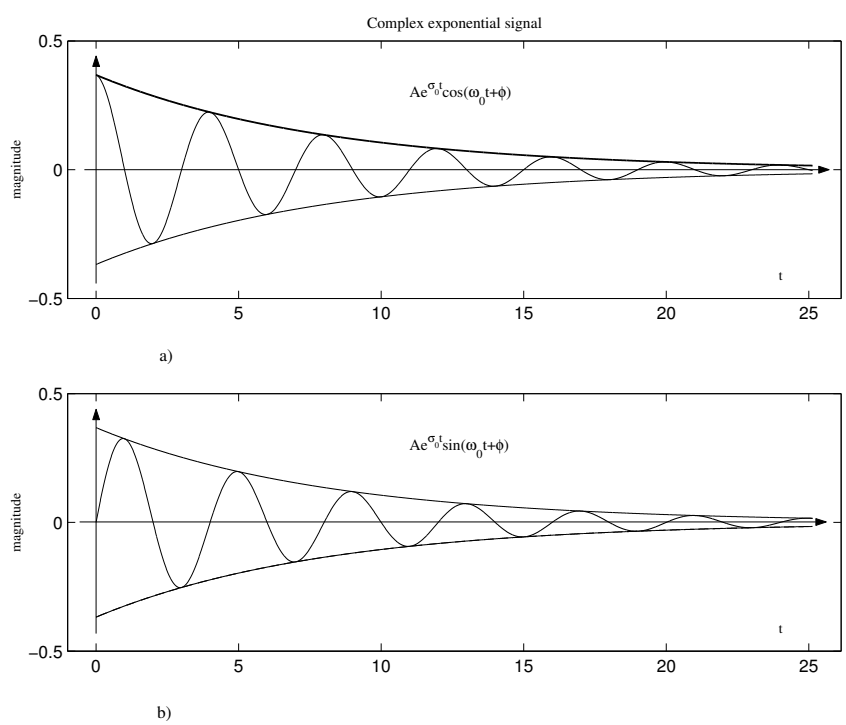


Fig. 43. Two complex exponential signals: cosine and sine waves in the exponential envelop.

VII. Unit Step Function

The unit step signal (or the Heaviside function) is defined as

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

Figure 44 shows this function in the interval $[-2, 5]$.

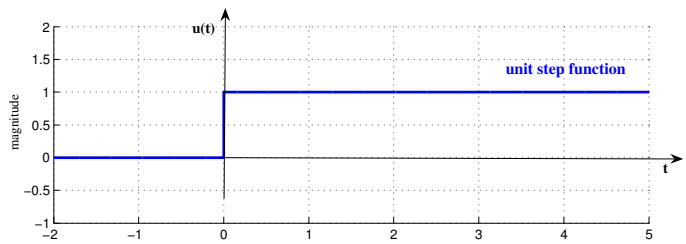


Fig. 44. The unit step function.

Given t_0, the time-shifted unit step signal is

$$u_{t_0}(t) = \begin{cases} 1, & t \geq t_0 \\ 0, & t < 0. \end{cases}$$

After the time transformation

$$t \to at - t_0, \quad a \neq 0,$$

the unit step signal changes as

$$\begin{aligned} u(t) &\to u\left(t - \frac{t_0}{a}\right), \quad a > 0 \\ &\to u\left(\frac{t_0}{a} - t\right), \quad a < 0. \end{aligned}$$

Indeed,[1]

a) if $a > 0$

$$\begin{aligned} u(at - t_0) &= u\left(a\left[t - \frac{t_0}{a}\right]\right) \\ &= \begin{cases} 1, & t - \frac{t_0}{a} \geq 0 \\ 0, & \text{otherwise} \end{cases} = u\left(t - \frac{t_0}{a}\right). \end{aligned}$$

b) if $a < 0$

$$u(at - t_0) = u\left(a\left[t - \frac{t_0}{a}\right]\right)$$

[1] Correction to (2.34) in the textbook

$$
\begin{aligned}
&= \begin{cases} 1, & t - \dfrac{t_0}{a} \le 0 \\ 0, & \text{otherwise} \end{cases} \\
&= u\left(-\left(t - \frac{t_0}{a}\right)\right) \\
&= u\left(\frac{t_0}{a} - t\right).
\end{aligned}
$$

A. Useful Properties of Unit function

a) switching $-1/+1$

$$x(t) = 2u(t) - 1$$

which is shown in Fig. 45.

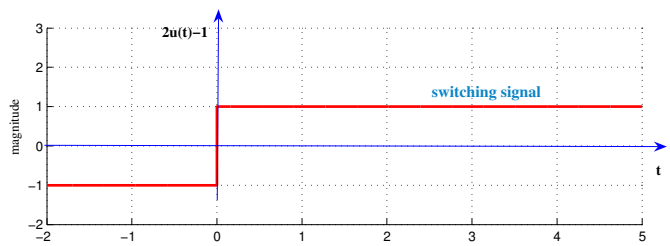


Fig. 45. The switching signal $2u(t) - 1$.

Example 1:

$$y(t) = x(t)\sin(t)$$

is a symmetric even function (see Fig. 46).

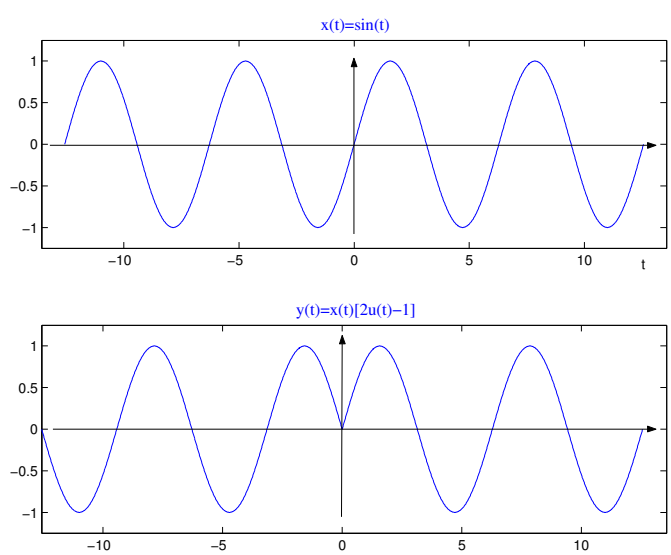


Fig. 46. The signal $x(t)\sin(t)$.

b) Unit pulse signal in $[T_1, T_2]$:
Let us consider two translations (time-shift transforms) of the unit step function

$$u_{T_1}(t) = u(t - T_1),$$

$$u_{T_2}(t) = u(t - T_2).$$

As an example, Figure 47 shows two such shifts for $T_1 = -1$ and $T_2 = 2$.

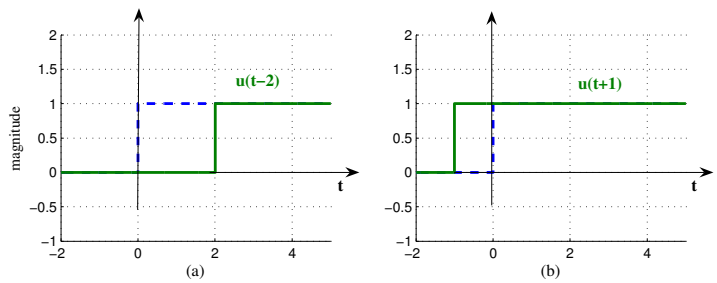


Fig. 47. The time-shifting transforms of the unit step function.

Assuming $T_1 < T_2$, we define

$$y(t) = u_{T_1}(t) - u_{T_2}(t) = \begin{cases} 1, & t \in [T_1, T_2] \\ 0, & \text{otherwise.} \end{cases}$$

as shown in Fig. 48, for $T_1 = -1$ and $T_2 = 2$.

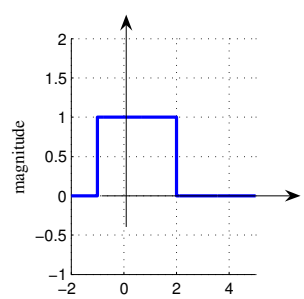


Fig. 48. The difference of two unit step functions (non symmetric rectangle).

c) Unit rectangular pulse is defined for the case when $T_1 = -T_2 = -0.5$,

$$\begin{aligned} rect(t) &= u_{-\frac{1}{2}}(t) - u_{\frac{1}{2}}(t) \\ &= \begin{cases} 1, & t \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The unit rectangular pulse in the symmetric interval $[-T/2, T/2]$ is defined as

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1, & -\frac{1}{2} \le \frac{t}{T} \le \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$rect\left(\frac{t}{T}\right) = \begin{cases} 1, & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$
$$= u_{-\frac{T}{2}}(t) - u_{\frac{T}{2}}(t).$$

d) Sum of shifted unit step signals.

Given $T > 0$, the unit step signal on the interval $[0, T/2]$ is defined as

$$u_{0,\frac{T}{2}}(t) = u_0(t) - u_{\frac{T}{2}}(t)$$

and the sum of shifted step signals

$$u_{0,\frac{T}{2}}(t) + u_{0,\frac{T}{2}}(t-T) + u_{0,\frac{T}{2}}(t-2T) + ... + u_{0,\frac{T}{2}}(t-nT)$$

is defined $(n+1)$ unit step signals.

The periodic parts of the considered above half-wave rectified signal can be defined as

$$v_1(t) = v_m \sin(\omega_0 t) u_{0,\frac{T}{2}}(t), \quad v_1(t-T), \quad v_1(t-2T), ...,$$

where $T = T_0 = 2\pi/\omega_0$.

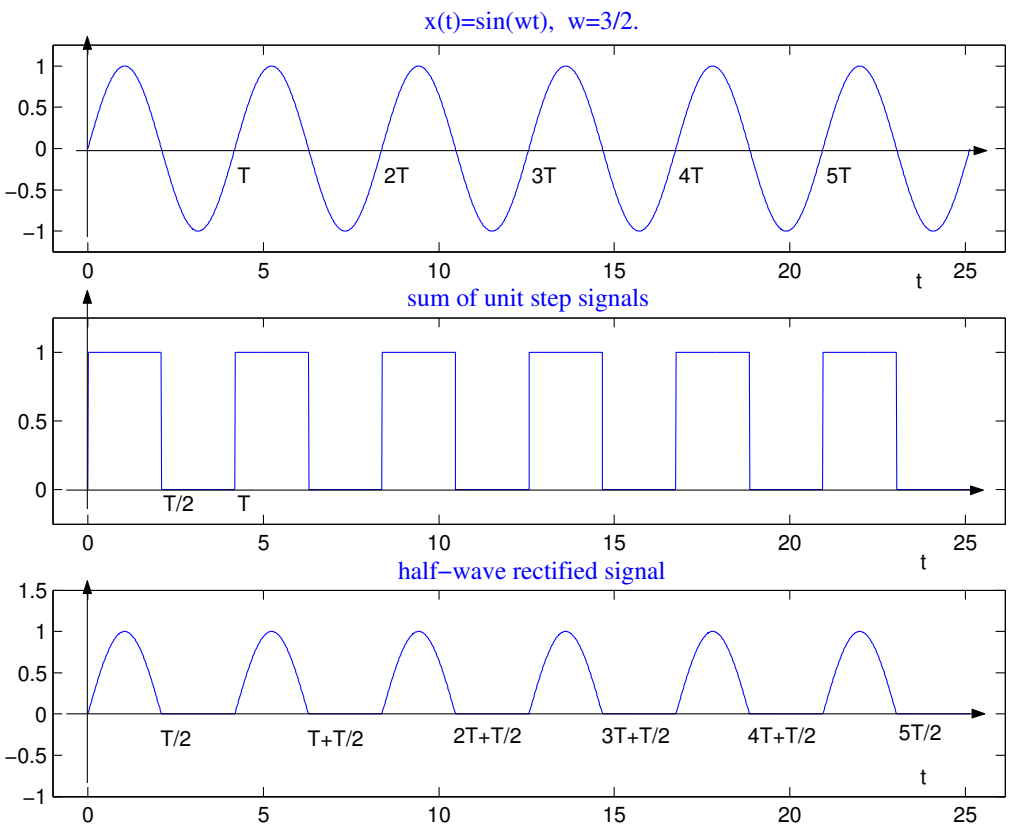


Fig. 49. The half-wave rectified signal.

B. ***The Unite Impulse Function*** *(δ-function)*

The delta function, $\delta(t)$, is a very important mathematical concept which is widely used in engineering. This function allows us to represent and describe many properties of physical systems in digital signal processing, including the filtration and restoration of signals.

Definition 1: $\delta(t)$ function (or Dirac delta function) is a generalized function, namely a functional which operates over functions continuous at point 0, $x(t)$, in the following way:

$$\delta : x \ \rightarrow \ x(0), \tag{38}$$

i.e., $\delta[x] = x(0)$, and this functional can be represented in the integral form as

$$\delta[x] = \int_{-\infty}^{\infty} x(t)\delta(t)dt,$$

where $\delta(t)$ is a function of t.

Thus, by the definition, the $\delta(t)$ function is a function such that

$$\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0), \tag{39}$$

<u>if the function $x(t)$ is continuous at the point $t = 0$.</u>

The property of the δ function:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{40}$$

There are many functions $g_n(t)$, $n = 1, 2, 3, ...$, exist such that the following convergence takes place

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x(t)g_n(t)dt = x(0). \tag{41}$$

In other words, we can say that $g_n(t) \rightarrow \delta(t)$. Sequence $g_n(t)$ is called delta-sequence.

Example 1:

$$g_n(t) = \begin{cases} 2^{n-1}, & \text{if } t \in \left[-\frac{1}{2^n}, \frac{1}{2^n}\right] \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

Figure **??** shows the first six functions $g_n(t)$.

Example 2: We also can consider the sequence of Gaussian functions

$$g_n(t) = \frac{1}{\sigma_n\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma_n^2}}, \qquad t \in (-\infty, +\infty), \tag{43}$$

for any sequence $\sigma_n \rightarrow 0$. For instance, $\sigma_1 = 2$, $\sigma_1 = 1$, $\sigma_1 = 0.5$, $\sigma_1 = 0.25$, ...

For all these functions, the integrals

$$\int_{-\infty}^{\infty} g_n(t)dt = 1 \tag{44}$$

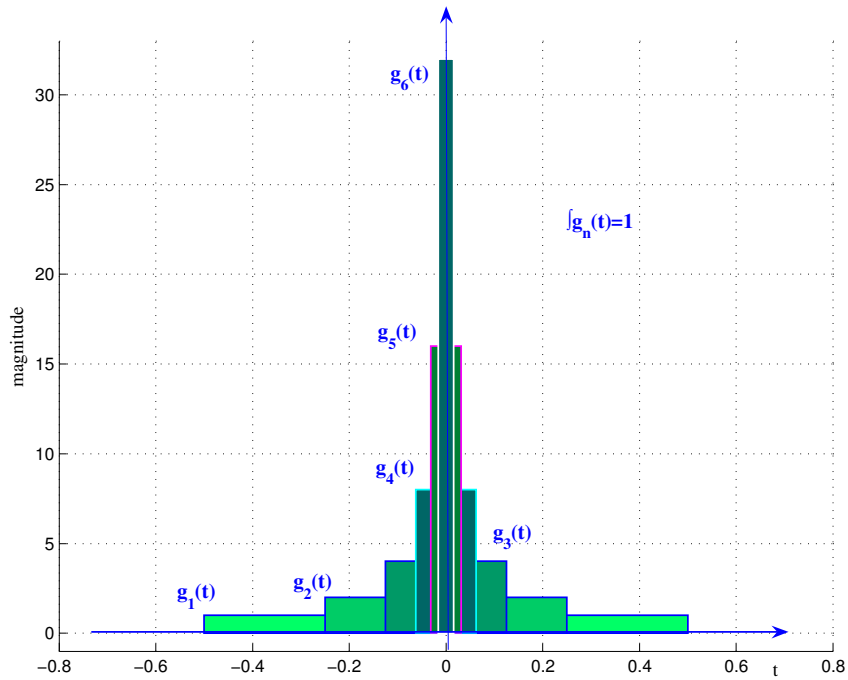


Fig. 50. Delta sequence of functions $g_n(t) \to \delta(t)$, as $n \to \infty$.

and (**??**) holds, since the following known result (Theorem of Mean) can be used

$$\int_{-\infty}^{\infty} x(t) g_n(t) dt = x(t_n)$$

for certain points $t_n \in [-1/2^n, 1/2^n]$ which approach to 0, when $n \to \infty$.

B.1 Properties of the δ-function

$$\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0).$$

1. Given $t_0 \neq 0$, the shifting $\delta(t - t_0)$ is defined as follows

$$\begin{aligned} \int_{-\infty}^{+\infty} f(t)\delta(t-t_0)dt &= (t - t_0 \to t) = \int_{-\infty}^{+\infty} f(t+t_0)\delta(t)dt \\ [g(t) &= f(t+t_0)] \\ &= \int_{-\infty}^{+\infty} g(t)\delta(t)dt = g(0) = f(t_0). \end{aligned}$$

So, if the function $f(t)$ is continuous at t_0, then

$$\int_{-\infty}^{+\infty} f(t)\delta(t-t_0)dt = f(t_0).$$

2. Given t_0,

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0).$$

To prove this equation, we have to use the correct (full) definition of the δ-function as a generalized function, i.e., *functional* - operation over the functions,

$$\delta : f \to f(0), \qquad \delta[f] = f(0).$$

Given a function $g(t)$, $t \in R$,

$$\begin{aligned}
\int_{-\infty}^{+\infty} g(t)\underline{f(t)\delta(t-t_0)}dt &= \int_{-\infty}^{+\infty} \underline{g(t)f(t)}\delta(t-t_0)dt \\
[c(t) &= g(t)f(t)] \\
\int_{-\infty}^{+\infty} c(t)\delta(t-t_0)dt &= c(t_0) = g(t_0)f(t_0) \\
&= \int_{-\infty}^{+\infty} g(t)\delta(t-t_0)dt \cdot f(t_0) \\
&= \int_{-\infty}^{+\infty} g(t)\underline{f(t_0)\delta(t-t_0)}dt
\end{aligned}$$

So, for any function $g(t)$ which is continuous at point t_0,

$$\int_{-\infty}^{+\infty} g(t)\underline{f(t)\delta(t-t_0)}dt = \int_{-\infty}^{+\infty} g(t)\underline{f(t_0)\delta(t-t_0)}dt$$

which results in the equality

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0).$$

3. Given $t_0 \neq 0$,

$$\int_{-\infty}^{t} \delta(\tau-t_0)d\tau = \int_{-\infty}^{+\infty} f(\tau)\delta(\tau-t_0)d\tau$$

$$\left(f(\tau) = \begin{cases} 1, & \tau \le t \\ 0, & \tau > t \end{cases} = \chi_{[-\infty,t]}(\tau) \right)$$

$$= f(t_0) = \begin{cases} 1, & t_0 \le t \\ 0, & t_0 > t \end{cases} = u(t-t_0)$$

So,

$$\int_{-\infty}^{t} \delta(\tau-t_0)d\tau = u(t-t_0).$$

4. The time transformation $t \to at - t_0$ of the delta function results

$$\delta(at-t_0) = \frac{1}{|a|}\delta\left(t - \frac{t_0}{a}\right), \quad a \neq 0, \tag{45}$$

which means that

$$\int_{-\infty}^{+\infty} f(t)\delta(at-t_0)dt = \frac{1}{|a|}\int_{-\infty}^{+\infty} f(t)\delta\left(t-\frac{t_0}{a}\right)dt$$

Indeed,

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(t)\underline{\delta(at-t_0)}dt &= (at-t_0 \to t',\ adt \to dt') \\
(t &= \frac{t'}{a}+\frac{t_0}{a},\ dt=\frac{1}{a}dt') \\
&= \frac{1}{a}\int_{\pm\infty}^{\mp\infty} f\left(\frac{t'}{a}+\frac{t_0}{a}\right)\delta(t')dt' \\
&= \frac{1}{|a|}\int_{-\infty}^{+\infty} f\left(\frac{t'}{a}+\frac{t_0}{a}\right)\delta(t')dt' \\
&= \frac{1}{|a|}f\left(\frac{0}{a}+\frac{t_0}{a}\right)=\frac{1}{|a|}f\left(\frac{t_0}{a}\right) \\
&= \frac{1}{|a|}\int_{-\infty}^{+\infty} f(t)\delta(t-\frac{t_0}{a})dt \\
&= \int_{-\infty}^{+\infty} f(t)\underline{\frac{1}{|a|}\delta(t-\frac{t_0}{a})}dt
\end{aligned}$$

5. Symmetry

$$\delta(-t)=\delta(t) \qquad \text{[as functional]}$$

Indeed, the time reversal transformation $t \to -t$ is a particular case of the time transformation $t \to at - t_0$ when $a=-1$ and $t_0=0$. Therefore, by property (**??**)

$$\delta(-t)=\delta(-1\cdot t-0)=\frac{1}{|-1|}\delta\left(t-\frac{0}{a}\right)=\delta(t).$$

In other words, for any function $f(t)$ which is continuous at 0

$$\int\limits_{-\infty}^{+\infty} f(t)\delta(-t)dt=\int\limits_{-\infty}^{+\infty} f(t)\delta(t)dt=f(0).$$

6. Time-scaling

$$\delta(at)=\frac{1}{|a|}\delta(t).$$

By substituting $t_0=0$, we obtain from property (**??**)

$$\delta(at)=\delta(at-0)=\frac{1}{|a|}\delta\left(t-\frac{0}{a}\right)=\frac{1}{|a|}\delta(t).$$

In other words, for any function $f(t)$ which is continuous at 0

$$\int\limits_{-\infty}^{+\infty} f(t)\delta(at)dt=\frac{1}{|a|}\int\limits_{-\infty}^{+\infty} f(t)\delta(t)dt=\frac{1}{|a|}f(0).$$

Example 3:
Find the value of the following integral:

$$\int_{-\infty}^{+\infty} [\cos^2(t^2+\pi/4)+1]\delta(t)dt.$$

Answer:

$$\int_{-\infty}^{+\infty} [\cos^2(t^2+\pi/4)+1]\delta(t)dt = [\cos^2(t^2+\pi/4)+1]_{|t=0} = \cos^2(\pi/4)+1 = \left(\frac{1}{\sqrt{2}}\right)^2+1 = \frac{1}{2}+1 = 1.5.$$

Example 4:
Find the value of the following integral:

$$\int_{-\infty}^{+\infty} [\cos^2(t^2+\pi/4)+1]\delta(2t-\frac{1}{4})dt.$$

Answer:

$$\int_{-\infty}^{+\infty} [\cos^2(t^2+\pi/4)+1]\delta(2t-\frac{1}{4})dt = \int_{-\infty}^{+\infty} [\cos^2(t^2+\pi/4)+1]\underline{\frac{1}{2}\delta(t-\frac{1}{8})}dt$$

$$= \frac{1}{2}[\cos^2(t^2+\pi/4)+1]_{|t=\frac{1}{8}} = \frac{1}{2}\cos^2(\frac{1}{64}+\pi/4)+1.$$

Example 5:
Evaluate the following integral:

$$\int_{-\infty}^{+\infty} [\sin(t-\pi/3)-1]\,\delta(3t-\frac{\pi}{3})dt.$$

Answer:

$$\int_{-\infty}^{+\infty} \left[\sin(t-\frac{\pi}{3})-1\right]\frac{1}{3}\delta(t-\frac{\pi}{9})dt = \frac{1}{3}\left[\sin(t-\frac{\pi}{3})-1\right]_{|t=\frac{\pi}{9}} = \frac{1}{3}\left[\sin(-\frac{2\pi}{9})-1\right] = -\frac{1}{3}\left[\sin(\frac{2\pi}{9})+1\right].$$

Example 6:
Evaluate the following integral:

$$\int_{-\infty}^{+\infty} e^{-2|t|}\delta(t-2)dt.$$

Answer:

$$\int_{-\infty}^{+\infty} e^{-2|t|}\delta(t-2)dt = e^{-2|t|}{}_{|t=2} = e^{-4}.$$

VIII. Systems

A *signal* is a function that represents the time (or coordinate) variation of a physical variable. When processing signals $x(t)$, we use the general concept of systems, which describe many physical systems such as a device, algorithm, filter. A system is a relationship, T, between two signals, $x(t)$ and $y(t)$, and can be defined as follows. A system generates a response (output signal $y(t)$) for a given input signal $x(t)$. The input represents a physical process that is generated independently from the system. The output is generated by the system when the input is present.

A system is represented by the mathematical symbol as (see Fig. 46)

$$y(t) = (T[x])(t), \quad \forall t \in R \quad (\text{or } t \in [a, b]),$$

which often is written as $y(t) = T[x(t)]$.

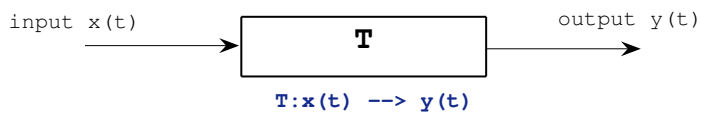


Fig. 46. Diagram of the system.

The process of deriving a system representation is called *modeling.*

A system which describe a relationship between continuous-time input $x(t)$ signals and continuous-time output $y(t)$ signals is called a *continuous-time* system.

A system which describe a relationship between discrete-time input $x(t)$ signals and discrete-time output $y(t)$ signals is called a *discrete-time* system.

If a system has one input and one output signal, then we call the system is a single-input-single-output (SISO) system. If a system has more than one input and/or output signal, then we call the system is a multi-input-multi-output (MIMO) system.

We have already considered the following simple systems:

$$\begin{array}{lll} T: x(t) \to x(-t), & \forall t, & \\ T: x(t) \to x(t - t_0), & t_0 \neq 0, & \text{(ideal time delay)} \\ T: x(t) \to x(at), & a \neq 0, & (\text{“compression”}) \\ T: x(t) \to Ax(t), & A \neq 0, & \text{(ideal amplifier).} \end{array}$$

In the $A = 1$ case, the system $T : x(t) \to x(t)$ is called the *identity* system.

We consider the important case, when a system H is described by the *linear convolution* which is calculated by the following integral

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau)d\tau, \quad t \in R. \tag{46}$$

$h(t)$ is the *impulse response* function of the system H. When the input signal is the ideal infinite impulse, i.e., "delta" function $\delta(t)$, or *unit impulse function* the output is

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)\delta(\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\delta(t - \tau)d\tau = h(t). \tag{47}$$

So, the impulse characteristics $h(t)$ of the system H can be determined as the output of the signal when input is the unit impulse function (ideal infinite impulse).

It is common to write (46) in the short form as

$$y = h * x, \qquad (y(t) = (h * x)(t) = h(t) * x(t).$$

This operation is commutative, $h * x = y = x * h$. The system H is also *linear,* which means that the following two conditions are satisfied for any inputs $x_1(t)$ and $x_2(t)$:

$$\begin{array}{l} H[x_1 + x_2] = H[x_1] + H[x_2], \\ H[Ax_1] = AH[x_1], \quad \text{for any constant } A. \end{array} \tag{48}$$

Indeed, taking any two functions $x_1(t)$ and $x_2(t)$, we can write the following for the linear convolution:

$$x_1(t) \to h(t) * x_1(t), \quad x_2(t) \to h(t) * x_2(t)$$

which implies

$$\int_{-\infty}^{\infty} h(t-\tau)[x_1(\tau) + x_2(\tau)]d\tau = \int_{-\infty}^{\infty} h(t-\tau)x_1(\tau)d\tau + \int_{-\infty}^{\infty} h(t-\tau)x_2(\tau)d\tau$$

or,

$$(x_1(t) + x_2(t) \to h(t) * (x_1(t) + x_2(t)) = h(t) * x_1(t) + h(t) * x_2(t).$$

Comments: In the discrete-time signal case, the linear convolution of the signal $x(n)$ with sequence $h(n)$ is defined as

$$\begin{aligned} y(n) &= \sum_{m=-\infty}^{\infty} h(n-m)x(m), \quad n \in Z = \{0, \pm 1, \pm 2, \ldots\}, \\ &= \sum_{m=-M}^{M} h(m)x(n-m). \end{aligned} \tag{49}$$

When $h(n) = 1/(2M+1)$, for $n = -M : M$, and $h(n) = 0$ for other points, the linear convolution results in the local means of the input signals in window $W = \{-M, ..., -1, 0, 1, ..., M\}$,

$$y(n) = x_{mean,W}(n) = \sum_{m=-M}^{M} h(m)x(n-m), \quad n \in Z = \{0, \pm 1, \pm 2, \ldots\}. \tag{50}$$

For instance, let the *impulse characteristic* of the system be is $h(m) = 1/3(1, \underline{1}, 1)$. This is so-called the mean filter with window 3, for which the output is calculated by

$$y(n) = \frac{1}{3}[x(n-1) + x(n) + x(n+1)].$$

As an example, Figure 47 shows the original signal degraded by a random impulse noise and the result of the linear convolution (mean filter). For comparison, the result of filtration with nonlinear median filter is also given. In general, the mean filter provides better filtration in the terms of the root mean square error (RME), not the mean-absolute error (MAE).

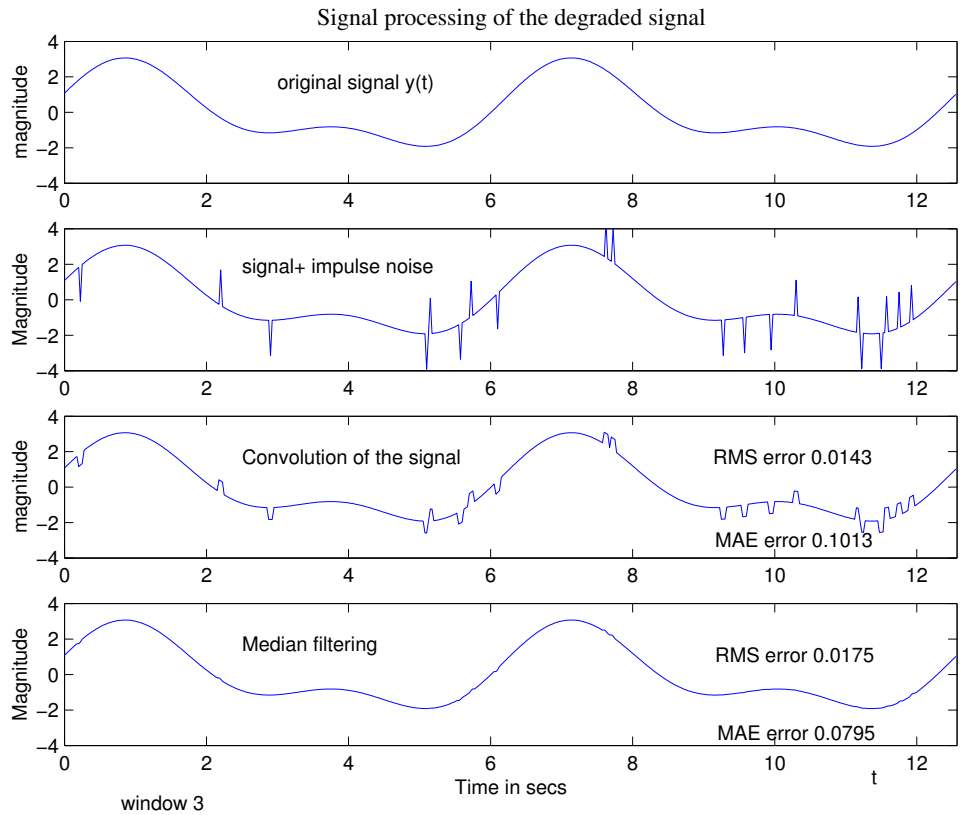


Fig. 47. Linear convolution of the degraded signal by mean filter with the symmetric window of length 3.

We can also consider other impulse responses. For instance, when $h(m) = 1/5(1, 1, \underline{1}, 1, 1)$, the convolution is calculated by

$$y(n) = \frac{1}{5}[x(n-2) + x(n-1) + x(n) + x(n+1) + x(n+2)],$$

and, in the $h(m) = 1/9(1, 2, \underline{3}, 2, 1)$ case, the output is calculated by

$$y(n) = \frac{1}{9}[x(n-2) + 2x(n-1) + 3x(n) + 2x(n+1) + x(n+2)],$$

for $n = 0, \pm 1, \pm 2, \ldots$.

The basic connections for systems are *parallel* and *cascade* connections which are shown in Figures 48 and 49:

$$T_1 : x(t) \to y(t), \quad T_2 : y(t) \to z(t), \quad z = T_2[T_1[x]] = (T_2 \circ T_1)[x],$$
$$T_1 : x(t) \to y_1(t), \quad T_2 : x(t) \to y_2(t), \quad (T_1 + T_2) : x(t) \to (y_1 + y_2)(t).$$

A. Properties of systems

1. A system is called *memoryless*, if an output $y(t)$ at any time t depends only on input of the signal $x(t)$ at the same time: $(T[x])(t) = T(x(t))$. Example: $y(t) = Ax(t) + B$. But the system

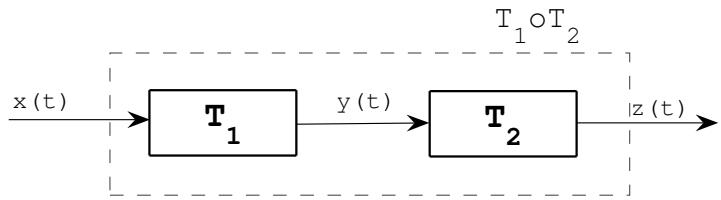


Fig. 48. Cascade-wise connections of two systems.

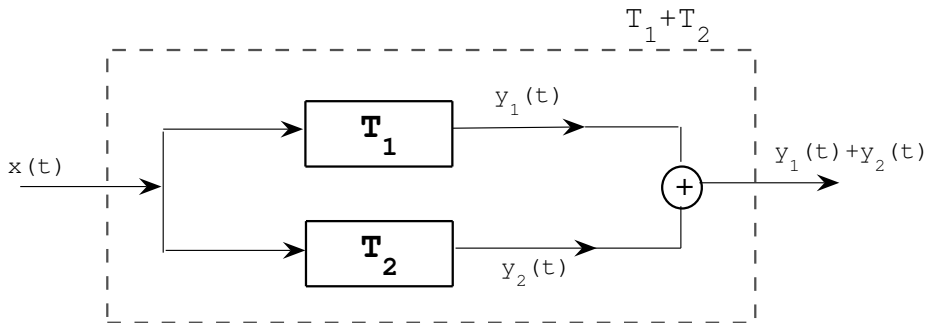


Fig. 49. Two systems are connected in parallel.

$y(t) = x(at - b)$ for $a \neq 1$ is not memoryless, as well as the system $y(t) = x(t) - x(t-1)$; we say such systems have memory.

2. A system is called *invertible*, if any inputs $x_1(t) \neq x_2(t)$ result in outputs $y_1(t) \neq y_2(t)$.

3. A system T_1 is called *inverse* to a system T, if $T_1 \circ T = I$, which means that $T_1 : \ y(t) \to x(t)$, when $T : \ x(t) \to y(t)$, for any input signals $x(t)$. The inverse system is denoted by T^{-1}. For example, if $T : \ x(t) \to y(t) = x(t-1)$, then the system $T^{-1} : \ y(t) \to y(t+1)$, because $y(t+1) = x(t)$ for all t. So $T^{-1} : \ y(t) \to x(t)$.

4. A system is called *causal*, if output $y(t_0)$ at any time t_0 depends only on input of the signal $x(t)$ at time $t \leq t_0$. For example, $y(t) = x(t) - x(t-1)$, but the system with outputs $y(t) = x(t+1) - x(t)$ is not causal.

5. A system is called *bounded-input bounded-output (BIBO)* stable, if exists a real number $B > 0$ such that

$$|y(t)| \leq B, \qquad \text{when} \quad |x(t)| \leq A,$$

and $A > 0$ is a real number. This conditions should hold for any bounded input of the system.

6. A system is called *time invariant* (TI), if $x_{t_0}(t) \to y_{t_0}(t)$ for any time shift t_0, when $x(t) \to y(t)$, i.e.

$$\text{if} \quad x(t) \to y(t), \forall t \in R, \quad \text{then} \quad x(t - t_0) \to y(t - t_0).$$

For example, relationship $y(t) = x(at)$ determines a TI system $T : \ x(t) \to y(t)$, but

$$T : \ x(t) \to y(t) = e^{-t} x(t), \quad t \in R,$$

is not TI system (take $x(t) = sin(t)$). Indeed, for this system we can write the following

$$T : \ x(t - t_0) \to y_1(t) = e^{-t} x(t - t_0), \quad t \in R,$$

but, when $t_0 \neq 0$,

$$y(t-t_0) = e^{-t+t_0}x(t-t_0) = e^{t_0}[e^{-t}x(t-t_0)] = e^{t_0}y_1(t) \neq y_1(t),$$

which shows that

$$\text{when} \quad x(t) \to y(t), \quad \text{then} \quad x(t-t_0) \not\to y(t-t_0).$$

In the case when a system is not TI, we say the system is *time varying*.

Example 1: Consider the following system:

$$T: \ x(t) \to y(t) = tx(t) + 1.$$

This system is not TI. To show that, consider the following response

$$T: \ x(t-1) \to tx(t-1) + 1$$

and that is not $y(t-1) = (t-1)x(t-1) + 1$.

This system is:

1. not linear
2. not invertible
3. not inverse
4. has memory
5. causal
6. not stable (not BIBO).

IX. Impulse Representation of Signals

In this section, we consider main properties of linear time-invariant systems. By definition of the delta function, $\delta(t)$, every continuous function (continuous-time continuous signal) $x(t)$ can be represented in the integral form as

$$\begin{aligned} x(t) &= \int_{-\infty}^{+\infty} \delta(t-\tau)x(\tau)d\tau = \int_{-\infty}^{+\infty} x(t-\tau)\delta(\tau)d\tau \\ &= \int_{-\infty}^{+\infty} \delta(\tau-t)x(\tau)d\tau \\ &= (\delta * x)(t) = \delta(t) * x(t). \end{aligned} \tag{51}$$

This identity convolution system is shown in Fig. 50.

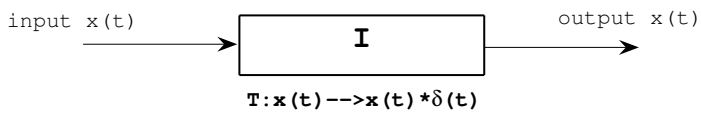


Fig. 50. Diagram of the identity system.

In general when a system, H, is linear and time-invariant, the output (response) of the system when input is $x(t)$ is described in the form of the linear convolution of the input with a function, which is called *the impulse response* function $h(t)$:

$$\begin{aligned} H : x(t) \to_{h(t)} y(t) &= \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)d\tau = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau \\ &\neq \int_{-\infty}^{+\infty} h(\tau-t)x(\tau)d\tau \quad \text{(if } h(\tau) \text{ is not symmetric)} \\ &= (h * x)(t) = h(t) * x(t) = x(t) * h(t) \end{aligned}$$

and therefore, $H : \delta(t) \to h(t)$ (see Fig. 51). The impulse response function is the response of the

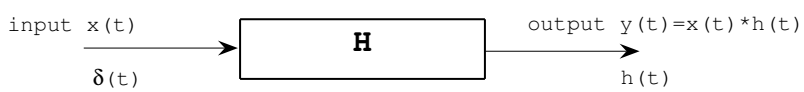


Fig. 51. Diagram of the LTI system with the impulse response $h(t)$.

system on the input being $\delta(t)$. The identity system of Fig. 50 has the impulse response function $h(t) = \delta(t)$.

Example 1 (Integrator) Consider a causal system H which is described as

$$H : x(t) \to y(t) = \int_{-\infty}^{t} x(\tau)d\tau, \quad t \in (-\infty, +\infty).$$

H is linear time invariant system, since for any given t_0, we have the following

$$H : x(t-t_0) \to \int_{-\infty}^{t} x(\tau - t_0)d\tau = \int_{-\infty}^{t-t_0} x(\tau')d\tau'$$

but

$$\int_{-\infty}^{t-t_0} x(\tau')d\tau' = y(t-t_0).$$

So, $H : x(t-t_0) \to y(t-t_0)$ when $x(t) \to y(t)$, for any t_0.

Example 2 (Impulse and Signal) Consider the linear convolution system H with the impulse response $h(t)$. Let $y(t)$ be the response of the system on the input signal

$$x(t) = \delta(t+3) + 3e^{-\frac{1}{2}t}u(t)$$

where $u(t)$ is the unit function. Since the system is linear, we can consider the input as the sum of two signals as follows:

$$x(t) = (x_1(t) = \delta(t+3)) + \Big(x_2(t) = 3e^{-\frac{1}{2}t}u(t)\Big)$$

and, then, compute two outputs

$$H : x_1(t) \to y_1(t)$$
$$H : x_2(t) \to y_2(t).$$

The sum $y_1(t) + y_2(t)$ is the output of the system when input is $x(t)$, i.e.,

$$H : x(t) \to H[x_1(t) + x_2(t)] = H[x_1(t)] + H[x_2(t)] = y_1(t) + y_2(t) = y(t).$$

It is clear, that

$$y_1(t) = \int_{-\infty}^{+\infty} h(t-\tau)x_1(\tau)d\tau = \int_{-\infty}^{+\infty} h(t-\tau)\delta(\tau+3)d\tau = h(t-\tau)_{|_{\tau=-3}} = h(t+3). \tag{52}$$

Let $h(t)$ be the rectangle $u(t) - u(t-2)$ on the interval $[0, 2]$. To compute the output $y_2(t)$, we first note that the area A of Fig. 52 is

$$A = A(t) = 2(1 - e^{-\frac{1}{2}t}).$$

Indeed

$$\int_{0}^{+\infty} e^{-t}dt = 1, \quad \text{therefore} \quad \int_{0}^{+\infty} e^{-\frac{1}{2}t}dt = 2\int_{0}^{+\infty} e^{-t}dt = 2.$$

We now write the convolution in the form

$$\int_{-\infty}^{+\infty} h(t-\tau)x_2(\tau)d\tau = \int_{-\infty}^{+\infty} h'(\tau - t)x_2(\tau)d\tau$$

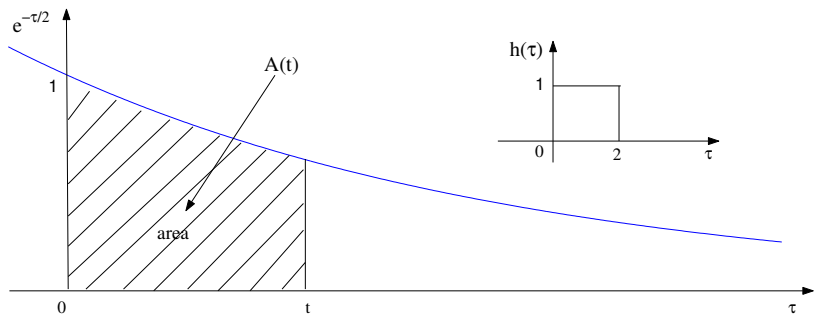


Fig. 52. The real exponential signal $e^{-\frac{1}{2}t}$.

where $h'(\tau) = h(-\tau)$.

1. For the case when $0 \in [t-2, t]$, i.e., $t \in [0, 2]$, we obtain

$$y_2(t) = \int_{-\infty}^{+\infty} h'(\tau - t) x_2(\tau) d\tau = \int_0^t 3e^{-\frac{1}{2}\tau} d\tau = 3A(t) = 6(1 - e^{-\frac{1}{2}t}).$$

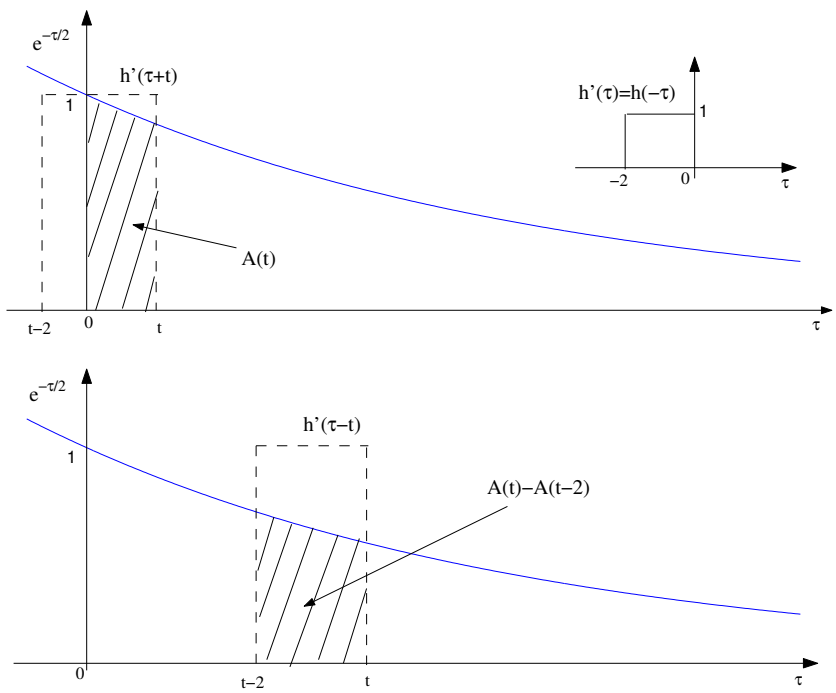


Fig. 53. The convolution of $(x_2 * h')(t)$.

2. For the case when $t > 2$, we obtain

$$\begin{aligned} y_2(t) &= \int_{-\infty}^{+\infty} h'(\tau - t)x_2(\tau)d\tau = \int_{t-2}^{t} h'(\tau - t)x_2(\tau)d\tau \\ &= A(t) - A(t-2) = 6(1 - e^{-\frac{1}{2}t}) - 6(1 - e^{-\frac{1}{2}(t-2)}) = 6(e^{-\frac{1}{2}(t-2)} - e^{-\frac{1}{2}t}) = 6(e-1)e^{-\frac{1}{2}t}. \end{aligned}$$

Therefore, the output of the linear system is

$$\begin{aligned} y(t) &= y_1(t) &&= [u(t+3) - u(t+1)] \\ &+ y_2(t) &&= [u(t) - u(t-2)]6(1 - e^{-\frac{1}{2}t}) + [u(t-2)]6(e-1)e^{-\frac{1}{2}t}. \end{aligned}$$

A. Properties of the Linear Convolution

The following properties of the linear convolution follow directly from the definition of the linear convolution

1. $x * h = h * x$
2. $x * (h_1 + h_2) = x * h_1 + x * h_2$
3. $(x_1 + x_2) * h = x_1 * h + x_2 * h$
4. $(x * h_1) * h_2 = x * (h_1 * h_2) = x * (h_2 * h_1) = (x * h_2) * h_1$
5. $x * (kh) = k(x * h) = (kx) * h, \qquad k \in R.$

A.1 Causality

The linear convolution can be written as follows:

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{0}^{+\infty} x(t-\tau)h(\tau)d\tau + \int_{-\infty}^{0} x(t-\tau)h(\tau)d\tau \\ &= \int_{0}^{+\infty} x(t-\tau)h(\tau)d\tau + 0 \text{ ("for system } H \text{ to be causal")} \\ &= \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau \end{aligned}$$

The second integral should be zero for all input signals $x(t)$, if the system H is causal, since the output $y(t)$ is a function of $\{x(t-\tau);\, \tau \geq 0\}$. Therefore, the impulse response function $h(-\tau) = 0$, for all $\tau > 0$. This is the only condition for a linear system to be *causal*.

A.2 Stability

In a stable linear system H, every bounded input results in a bounded output, i.e., if

$$|x(t)| \leq M, \quad \forall t \in R$$

then, $\exists N > 0$ such that

$$|y(t)| \leq N, \quad \forall t \in R, \quad \text{if} \quad H : x(t) \to y(t).$$

To show the sufficiency of this condition for stability of the system, let us take the following function

$$x(t) = \begin{cases} +1; & \text{if } h(-t) \geq 0, \\ -1; & \text{if } h(-t) < 0. \end{cases}$$

Then, $|x(t)| \equiv 1$, i.e., $x(t)$ is bounded, and at point $t = 0$ we obtain

$$\int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} x(-\tau)h(\tau)d\tau = \int_{-\infty}^{+\infty} |h(\tau)|d\tau$$

In order that L to be stable, the last integral should exist, i.e.,

$$\int_{-\infty}^{+\infty} |h(\tau)|d\tau = B < \infty.$$

This is the necessary and also sufficient condition for stability of the system. Indeed, if such condition takes place, then, for a bounded function $|x(t)| \leq M$,

$$|y(t)| \quad = \quad \left| \int_{-\infty}^{+\infty} x(\tau - t)h(\tau)d\tau \right| \leq \int_{0}^{+\infty} |x(\tau - t)||h(\tau)|d\tau \leq M \int_{0}^{+\infty} |h(\tau)|d\tau < MB, \quad \forall\, t.$$

Example 3: Consider the impulse response function $h(t) = e^{-at}u(t), \quad a > 0$. Then

$$\int_{-\infty}^{+\infty} |h(t)|dt = \int_{0}^{+\infty} h(t)dt = \frac{1}{a}\int_{0}^{+\infty} e^{-at}d(at) = -\frac{1}{a}e^{-t}\Big|_{0}^{\infty} = \frac{1}{a}$$

and the system is *stable*.

In the case, when the impulse response function $h(t) = e^{at}u(t), \quad a > 0$, we obtain

$$\int_{-\infty}^{+\infty} |h(t)|dt = \int_{0}^{+\infty} h(t)dt = \frac{1}{a}e^{t}\Big|_{0}^{\infty} = \infty$$

and the system is not stable.

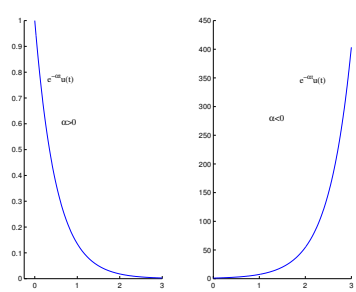


Fig. 54. Two exponential signals $e^{at}u(t)$, when $a > 0$ and $a < 0$.

X. LTI Systems and Differential Equations

Many physical systems as linear time-invariant systems are defined by differential equations with constant coefficients

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + a_2 \frac{d^{n-2} y(t)}{dt^{n-2}} + \cdots + a_n y(t) = b_0 \frac{d^m x(t)}{dt^m} + b_1 \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_n x(t),$$

which can also be written as

$$L_y(1, a_1, a_2, \ldots, a_n) = L_x(b_0, b_1, b_2, \ldots, b_m). \tag{53}$$

Here, integers $n \geq m$, and a_k, $k = 1 : n$ and b_k, $k = 0 : m$, are constant coefficients; n is the order of the differential equation.

What we need to know:

1. The above equation can be considered as

$$L_y(1, a_1, a_2, \ldots, a_n) = L_x(b_0, b_1, b_2, \ldots, b_m) + 0.$$

Therefore, the solution of this equation $y(t)$ can be represented as the solution $y_0(t)$ of the homogeneous equation

$$L_y(1, a_1, a_2, \ldots, a_n) = 0,$$

and single solution y_1 of the equation

$$L_y(1, a_1, a_2, \ldots, a_n) = L_x(b_0, b_1, b_2, \ldots, b_m).$$

The general form of the solution of the equation in (53) is

$$y(t) = y_1(t) + y_0(t).$$

To make this solution unique, we need to know n initial values (conditions), for instance values

$$y(0),\ y^{(1)}(0),\ y^{(2)}(0),\ \ldots,\ y^{(n-1)}(0)$$

at the point $t = 0$ or any other point $t = t_0$.

2. For the solution $y_0(t)$ of the homogeneous equation

$$L_y(1, a_1, a_2, \ldots, a_n) = 0,$$

we study the corresponding characteristic polynomial

$$P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1} \lambda + a_n$$

and solve the characteristic equation $P(\lambda) = 0$. Assuming all roots of this equations are different, $\lambda_1, \lambda_2, \ldots, \lambda_n$, the solution of the homogeneous equation can be written as

$$y_0(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \cdots + C_n e^{\lambda_n t},$$

where the coefficients C_k are found from the initial conditions (they should be given).

3. If one of the roots, let say λ_1 is complex, then the complex conjugate $\bar{\lambda}_1$ is also the root of the characteristic polynomial, i.e., $P(\bar{\lambda}_1) = 0$. So, $\bar{\lambda}_1$ is one of the roots, let say λ_3.

It also means that if the equation is real (or system is real, i.e., real input $\to$ real output), then the solution $y_0(t)$ contains the cosine wave,

$$C_1 e^{\lambda_1 t} + C_3 e^{\lambda_3 t} = C_1 e^{\lambda_1 t} + C_3 e^{\bar{\lambda}_1 t}$$

and $C_3 = \bar{C}_1$ (since the system is real), therefore,

$$C_1 e^{\lambda_1 t} + C_3 e^{\lambda_3 t} = C_1 e^{\lambda_1 t} + \bar{C}_1 e^{\bar{\lambda}_1 t} = C_1 e^{\lambda_1 t} + \overline{C_1 e^{\lambda_1 t}}.$$

C_1 and λ_1 in general is are complex numbers, $C_1 = |C_1| e^{i\varphi}$ and $\lambda_1 = \sigma_1 + i\omega_1$. Therefore, we can continue the above equation and write

$$C_1 e^{\lambda_1 t} + \overline{C_1 e^{\lambda_1 t}} = |C_1| e^{i\varphi} e^{(\sigma_1 + i\omega_1)t} + |C_1| e^{-i\varphi} e^{(\sigma_1 - i\omega_1)t} = 2|C_1| e^{\sigma_1 t} \cos(\omega_1 t + \varphi).$$

A. First-order LTI systems

We consider linear time-invariant systems, which are defined by the simple differential equation

$$\frac{dy(t)}{dt} - ay(t) = x(t), \quad a > 0,$$

that describe RL circuits ($y(t) = i(t)$, $x(t) = v(t)$).

Assuming the input signal is constant, the above equation takes the form

$$\frac{dy(t)}{dt} + ay(t) = const, \quad \forall t. \tag{54}$$

It is clear that the constant function $y_0(t) = const/a$ is a solution (named a *particular* solution) of this equation. Therefore, we find the solution of the differential equation

$$\frac{dy_1(t)}{dt} + ay_1(t) = 0. \tag{55}$$

and define the solution of Eq. 54 as $y(t) = y_1(t) + y_0(t)$.

We will find a solution of Eq. 55 in the form of the exponential function,

$$y_1(t) = Ce^{-st}$$

where s is real number.

Substituting $y(t)$ in Eq. 55, we obtain

$$-sCe^{-st} + aCe^{-st} = 0, \quad \forall t \in (-\infty, +\infty),$$

which shows that $s = a$, and the solution

$$y(t) = Ce^{-at} + \text{const}/a.$$

The value of C can be found from $y(0)$,

$$y(0) = Ce^{-a0} + \text{const}/a = C + \text{const}/a,$$

or $C = y(0) - \text{const}/a$.

B. General solution of the 1st order differential equation

In this section, we drive the analytical formula for the solution of the differential equation differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), \tag{56}$$

with a constant coefficient a and initial value $y(t_0)$. [1]

For that, first we define the function

$$f(t) = e^{at}y(t) \tag{57}$$

and consider the derivative of this (unknown) function

$$\frac{df(t)}{dt} = e^{at}\frac{dy(t)}{dt} + ae^{at}y(t) = e^{at}\left[\frac{dy(t)}{dt} + ay(t)\right] = e^{at}x(t).$$

Therefore, we can write the following (when $t \geq t_0$):

$$f(t) = \int_{-\infty}^{t} e^{a\tau}x(\tau)d\tau = f(t_0) + \int_{t_0}^{t} e^{a\tau}x(\tau)d\tau.$$

Also,

$$f(t_0) = e^{at_0}y(t_0)$$

and

$$f(t) = e^{at_0}y(t_0) + \int_{t_0}^{t} e^{a\tau}x(\tau)d\tau.$$

From Equation 57 it follows that

$$y(t) = e^{-at}f(t) = e^{-at}\left[e^{at_0}y(t_0) + \int_{t_0}^{t} e^{a\tau}x(\tau)d\tau\right] = e^{-a(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{-a(t-\tau)}x(\tau)d\tau$$

or

$$y(t) = e^{-a(t-t_0)}y(t_0) + e^{-at}\int_{t_0}^{t} e^{a\tau}x(\tau)d\tau. \tag{58}$$

This is the general solution of the differential equation (57).

Example 1: Consider the RC circuit between two notes across which the voltage potential is $x(t)$. Let R be of 1Ω and C be of 1/4F. The voltage across the capacitor is described by the following differential equation:

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

or

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

[1] You can omit the proof and consider only Eq. 58

We consider that the voltage $x(t) = 2\text{V}$, namely $x(t) = 2u(T)$ and the initial value at $t_0 = 0$ is $y(0) = 0$.

As follows from (58), the following calculations hold for the solution of th RC system:

$$a = 1/RC, \quad \text{and} \quad x(t) = x(t)/(RC) = 2u(t)/(RC)$$

and (when $t \geq 0$)

$$\begin{aligned} y(t) &= e^{-a(t-0)}0 + e^{-at}\int_0^t e^{a\tau}2u(\tau)d\tau = e^{-\frac{1}{RC}t}\int_0^t e^{\frac{1}{RC}\tau}2u(\tau)\frac{1}{RC}d\tau \\ &= 2u(t)e^{-\frac{1}{RC}t}\int_0^t de^{\frac{1}{RC}\tau} = 2u(t)e^{-\frac{1}{RC}t}\left[e^{\frac{1}{RC}t} - 1\right] = 2u(t)\left[1 - e^{-\frac{1}{RC}t}\right]. \end{aligned}$$

The time constant $RC = 1/4$ and $1/(RC) = 4$, and the final answer for the solution is

$$y(t) = 2\left[1 - e^{-4t}\right]u(t).$$

C. Transfer function for the 1-O LTIS

We now define a concept of a *transfer function* for the linear system H that is defined by the differential equation

$$\frac{dy(t)}{dt} - ay(t) = x(t), \quad a > 0, \tag{59}$$

Given a value $s \in R$, we consider the input $x(t)$ as a exponential function

$$x(t) = x_s(t) = Xe^{st}, \quad t \in (-\infty, +\infty), \ X \neq 0,$$

and output of the system

$$y(t) = y_s(t) \ \leftarrow \ x_s(t) = x(t).$$

We now consider the output $y(t)$ in the form

$$y(t) = y_s(t) = \mathbf{H}[x_s(t)] = H(s)x_s(t) = H(s)x(t),$$

where $H(s) \neq 0$ is an unknown (for the present) function to be defined.

Owing to Eq. 59, we have the following:

$$\frac{d(H(s)x(t))}{dt} - a(H(s)x(t)) = x(t),$$

$$H(s)\frac{dx(t)}{dt} - (aH(s))x(t) = x(t),$$

$$H(s)\frac{dx(t)}{dt} = (aH(s) + 1)x(t).$$

The last equality can be written as

$$\frac{dx(t)}{dt} = C(s)x(t), \tag{60}$$

where $C(s) = a + H(s)^{-1}$ when $H(s) \neq 0$.

Because of our assumtion $x(t) = Xe^{st}$, the equation

$$\frac{dx(t)}{dt} = C(s)x(t),$$

yields

$$\frac{dx(t)}{dt} = Xse^{st} = C(s)Xe^{st}$$

and for $X \neq 0$,

$$C(s) = s, \quad \text{or}, \quad a + H(s)^{-1} = s$$

Therefore, the function

$$H(s) = \frac{1}{s-a}, \quad \text{if} \quad s \neq a.$$

We define $H(s)$ to be a *transfer function of 1st order* of the system H defined by the first-order differential equation

$$\frac{dy(t)}{dt} - ay(t) = x(t), \quad a > 0,$$

and for which the following transformation takes place

$$x(t) \to y(t) = \frac{1}{s-a}x(t). \tag{61}$$

We now define the same linear system H by using the integral-type linear convolution

$$y(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau$$

for a *response function* $h(t)$ to be found.

According to (61), we will define the response function in a way such that for the input function $x(t) = Xe^{st}$ the output $y(t)$ to be equal $\frac{1}{s-a}x(t)$.

By simple calculations, we have the following:

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} Xe^{s(t-\tau)}h(\tau)d\tau \\ &= Xe^{st}\int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau \\ &= x(t)\int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau \end{aligned}$$

Therefore, the transfer function of the system H equals

$$H(s) = \int_{-\infty}^{+\infty} e^{-s\tau} h(\tau) d\tau. \tag{62}$$

This equation shows the relation between the response function $h(t)$ of the system H and its transfer function $H(s)$.

D. n-order LTI systems

We now consider a general nth-order linear system $H : x(t) \to y(t)$ that is described by the n-order linear differential equation with constant coefficients,

$$a_0 y(t) + a_1 y^{(1)}(t) + a_2 y^{(2)}(t) + \ldots + a_n y^{(n)}(t) = b_0 x(t) + b_1 x^{(1)}(t) + b_2 x^{(2)}(t) + \ldots + b_m x^{(m)}(t) \tag{63}$$

with coefficients a_k and b_l, $(k = 0 : n,\ l = 0 : m)$. $y^{(k)}(t)$ and $x^{(l)}(t)$ denote respectively the derivatives $d^k y(t)/dt^k$ and $d^l x(t)/dt^l$. We consider an exponential input function with amplitude X,

$$x(t) = X e^{st}, \quad t \in (-\infty, +\infty)$$

and output in the form

$$y(t) = H(s) x(t), \quad t \in (-\infty, +\infty)$$

where $H(s)$ is a function of s.

Since

$$x^{(l)}(t) = s^l x(t) = s^l e^{st}, \quad l = 0, \ldots, m,$$

and

$$y^{(k)}(t) = H(s) x^{(k)}(t) = H(s) s^k x(t) = s^k y(t)$$

for k=0,...,n, we can write Eq. 63 as

$$\begin{aligned} a_0 y(t) + a_1 s y(t) + a_2 s^2 y(t) + \ldots + a_n s^n y(t) = \\ = b_0 x(t) + b_1 s x(t) + b_2 s^2 x(t) + \ldots + b_m s^m x(t) \end{aligned} \tag{64}$$

or,

$$[a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n] y(t) = [b_0 + b_1 s + b_2 s^2 + \ldots + b_m s^m] x(t). \tag{65}$$

We finally obtain

$$y(t) = \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n} x(t). \tag{66}$$

Therefore the nth-order linear system $H : x(t) \to y(t)$, that has been defined by the n-order linear differential equation with constant coefficients, is described by the *transform function*

$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n} \tag{67}$$

when inputs are exponential functions.

We now consider the linear convolution

$$y(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau$$

when input is $x(t) = e^{st}$.

$$y(t) = \int_{-\infty}^{+\infty} e^{s(t-\tau)}h(\tau)d\tau = e^{st}\int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau.$$

Also, the following holds:

$$y(t) = H(s)x(t) = e^{st}H(s), \quad t \in (-\infty, +\infty).$$

Therefore, we obtain the following formula for computing the response function of the system by the impulse function:

$$H(s) = \int_{-\infty}^{+\infty} e^{-s\tau}h(\tau)d\tau. \tag{68}$$

Generalizing the above reasoning, we see that the linear convolution system can be represented by two ways:

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \quad \text{(integral representation)},$$

$$x(t) \longrightarrow \boxed{H(s)} \longrightarrow y(t) \quad \text{(polynomial representation)}.$$

and the relations between these two functions is described by (68), which in the general case leads to the concepts of the integral Fourier and Laplace transforms.

XI. Block diagrams for LTI systems

We have defined the transfer function of the linear time-invariant systems which are described by differential equations. In particular case of the first-order differential equation

$$\frac{dy(t)}{dt} - ay(t) = x(t), \quad a > 0, \tag{69}$$

the function $H(s)$ is defined such that

$$x(t) = x_s(t) = Xe^{st} \to H(s)Xe^{st} = H(s)x(t)$$

and equals

$$H(s) = \frac{1}{s-a}.$$

Since, for a given t,

$$\int_{-\infty}^{t} \Big(\frac{dy(\tau)}{d\tau}\Big) d\tau = y(t)$$

Eq. 69, i.e.,

$$\frac{dy(t)}{dt} = ay(t) + x(t)$$

can be considered as

$$y(t) = \int_{-\infty}^{t} \Big[ay(\tau) + x(\tau)\Big] d\tau. \tag{70}$$

According to Eq. 70, the diagram of realization of the first-order LTI system takes the form which is shown in Fig. 55.

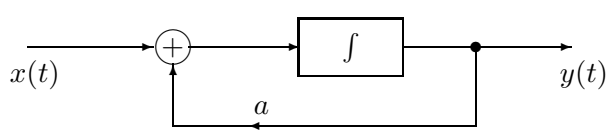


Fig. 55. Realization of the first-order LTI system of Eq. 69. (One integrator and one multiplier by factor a are used in the diagram.)

The block with $\int$ denotes the integrator,

$$x(t) \to y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

The response function, $h(t)$, of the integrator is the unit step function, $h(t) = u(t)$. Indeed, when the input function is $x(t) = \delta(t)$,

$$x(t) = \delta(t) \to h(t)$$

and

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{-\infty}^{+\infty} [1-u(\tau-t)]\delta(\tau)d\tau \\ &= [1-u(\tau-t)]_{|\tau=0} = 1-u(-t) = u(t). \end{aligned}$$

We can use also the known property of the δ function:

$$\int_{-\infty}^{t} \delta(\tau-t_0)d\tau = u(t-t_0)$$

for $t_0 = 0$.

So, we have three different forms of representation of the system as shown in Fig. 56. The first two forms hold for any inputs, and the last form of realization holds only for inputs having exponential form, i.e., for $x(t) = x_s(t) = Xe^{st}$, where X and s are constants.

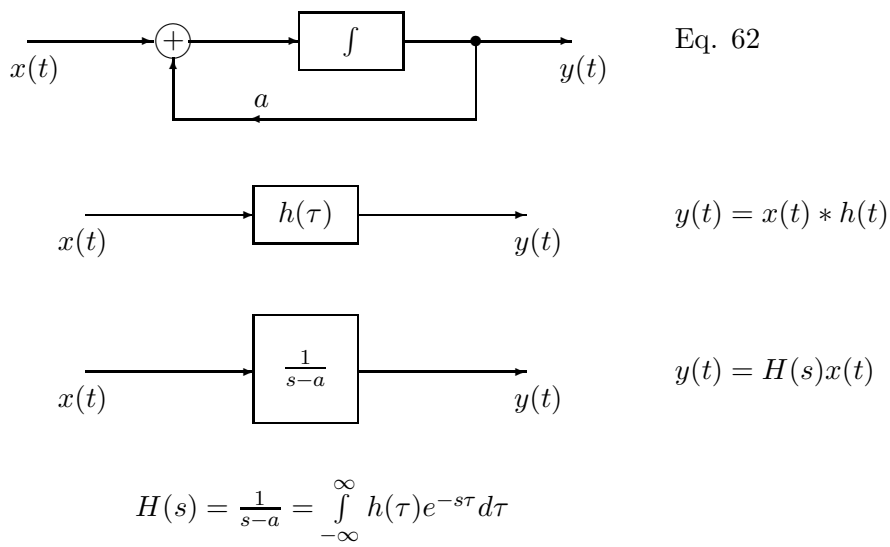


Fig. 56. Three diagrams of realization of the 1st order LTI system.

Remark: The first two systems result in the same outputs for equal input functions. The third system gives the similar outputs only if the input function is $x(t) = Xe^{st}$.

In the general case of the first-order differential equation

$$a_1 \frac{dy(t)}{dt} - ay(t) = bx(t) \tag{71}$$

where a_1, a, and b are some coefficients, we can write the realization of the system $x(t) \to y(t)$ as

$$y(t) = \frac{1}{a_1} \int_{-\infty}^{t} \Big[ay(\tau) + bx(\tau)\Big] d\tau \tag{72}$$

The diagram of realization of the first-order LTI system will take form shown in Fig. 57.

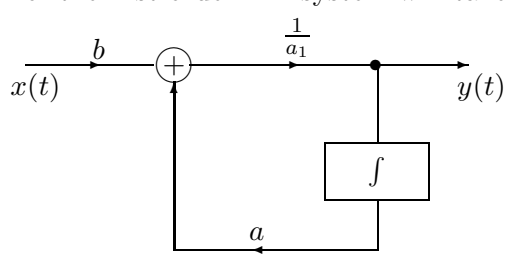


Fig. 57. (Is is correct?) Realization of the 1st order LTI system of Eq. 71.

We now consider the second-order differential equation

$$a_2 \frac{d^2y(t)}{dt^2} - a_1 \frac{dy(t)}{dt} - ay(t) = bx(t) \tag{73}$$

with coefficients a_2, a_1, a, and b. The realization of the system $x(t) \to y(t)$ can be represented as

$$a_2 \frac{d^2y(t)}{dt^2} = a_1 \frac{dy(t)}{dt} + ay(t) + bx(t)$$

$$\begin{aligned} a_2 \frac{dy(t)}{dt} &= \int_{-\infty}^{t} \Big[a_1 \frac{dy(\tau)}{d\tau} + ay(\tau) + bx(\tau)\Big] d\tau \\ &= a_1 y(t) + \int_{-\infty}^{t} \Big[ay(\tau) + bx(\tau)\Big] d\tau \end{aligned} \tag{74}$$

$$
\begin{aligned}
a_2 y(t) &= \int_{-\infty}^{t} a_1 y(\tau) d(\tau) + \int_{-\infty}^{t} \Big(\int_{-\infty}^{\tau} \Big[a y(\tau_1) + b x(\tau_1) \Big] d\tau_1 \Big) d\tau \\
y(t) &= \frac{1}{a_2} \Bigg[a_1 \int_{-\infty}^{t} y(\tau) d(\tau) + \int_{-\infty}^{t} \Big(\int_{-\infty}^{\tau} \Big[a y(\tau_1) + b x(\tau_1) \Big] d\tau_1 \Big) d\tau \Bigg] \\
& \frac{1}{a_2} \Bigg[b \int_{-\infty}^{t} \Big(\int_{-\infty}^{\tau} x(\tau_1) d\tau_1 \Big) d\tau + \\
& \qquad + a_1 \int_{-\infty}^{t} y(\tau) d\tau + a \int_{-\infty}^{t} \Big(\int_{-\infty}^{\tau} y(\tau_1) d\tau_1 \Big) d\tau \Bigg]
\end{aligned}
\tag{75}
$$

The diagram of realization of the first-order LTI system will take form shown in Fig. 58, where two integrators are used when input is $x(t)$ and two integrators are used when input is $y(t)$.

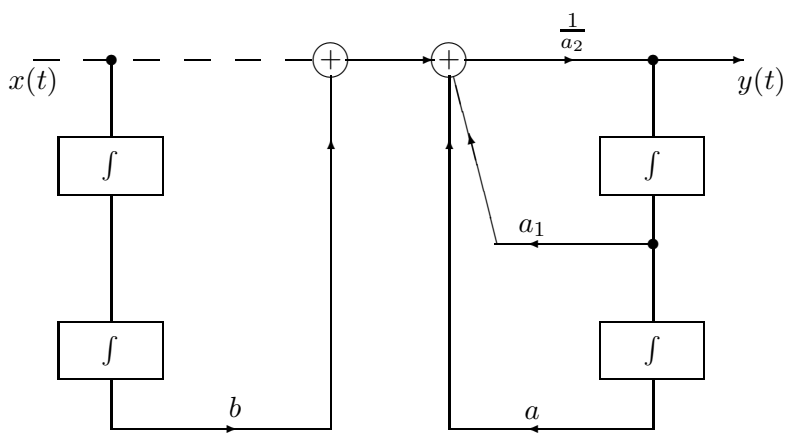


Fig. 58. Realization of the 2nd order LTI system of Eq. 75.

We consider the first-order differential equation of the form

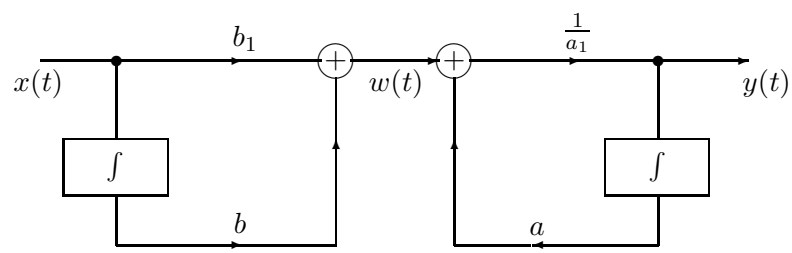
$$
a_1 \frac{dy(t)}{dt} - a y(t) = b x(t) + b_1 \frac{dx(t)}{dt} \tag{76}
$$

with coefficients a_1, a, and b, b_1

Similar to Eq. 72, we can write the realization of the system $x(t) \to y(t)$ as

$$\begin{aligned} y(t) &= \frac{1}{a_1}\int_{-\infty}^{t}\Big[ay(\tau)+bx(\tau)+b_1\frac{dx(\tau)}{d\tau}\Big]d\tau \\ &= \frac{1}{a_1}\Big(\int_{-\infty}^{t}\Big[ay(\tau)+bx(\tau)\Big]d\tau+b_1x(t)\Big) \\ &= \frac{1}{a_1}\Big(\int_{-\infty}^{t}ay(\tau)d\tau+b\int_{-\infty}^{t}x(\tau)d\tau+b_1x(t)\Big) \end{aligned} \tag{77}$$

The diagram of realization of that kind of first-order LTI system will take form shown in Fig. 59.



$$w(t) = b\int_{-\infty}^{t}x(\tau)d\tau + b_1x(t)$$

Fig. 59. Realization of the 1st order LTI system of Eq. 76.

In the case of the second-order differential equation of the form

$$a_2\frac{d^2y(t)}{dt^2} - a_1\frac{dy(t)}{dt} - ay(t) = bx(t) + b_1\frac{dx(t)}{dt}, \tag{78}$$

the realization of the system $x(t) \to y(t)$ can be represented as

$$\begin{aligned} y(t) &= \frac{1}{a_2}\Bigg[\int_{-\infty}^{t}a_1y(\tau)d(\tau) \qquad (a_2 \neq 0) \\ &\qquad + \int_{-\infty}^{t}\Big(\int_{-\infty}^{\tau}\Big[ay(\tau_1)+bx(\tau_1)+b_1\frac{dx(\tau_1)}{d\tau_1}\Big]d\tau_1\Big)d\tau\Bigg] \\ &= \frac{1}{a_2}\Bigg[\int_{-\infty}^{t}\Big(a_1y(\tau)+\int_{-\infty}^{\tau}ay(\tau_1)d\tau_1+ \\ &\qquad +b_1x(\tau)+\int_{-\infty}^{\tau}bx(\tau_1)d\tau_1\Big)d\tau\Bigg] \end{aligned} \tag{79}$$

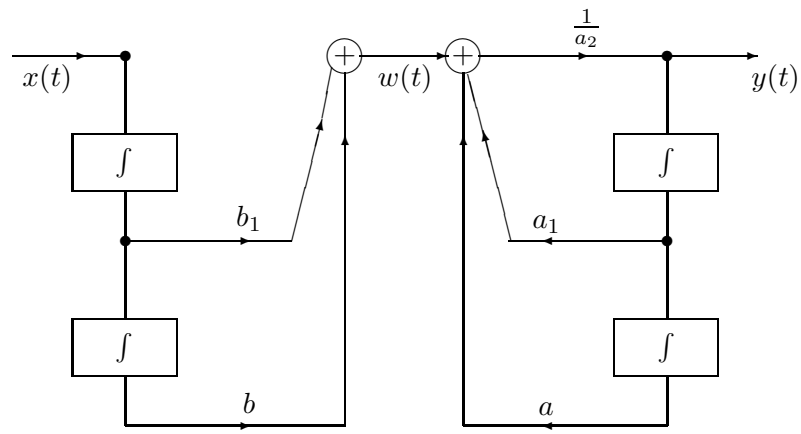


Fig. 60. Realization of the 2nd order LTI system of Eq. 78. Three integrators are required for this realization, but it will be shown later that the realization of this LTI system can be reduced to a diagram with two integrators. All described above diagrams are so-called diagram of realization of form I.

The diagram of realization of that kind of second-order LTI system will take form shown in Fig. 60.

The function $\omega(t) = b(x * u)(t) + b_1 x(t)$, i.e.,

$$\omega(t) = b \int_{-\infty}^{t} x(\tau) d\tau + b_1 x(t),$$

because the unit step function $u(t)$ is the impulse response function of the integrator.

Eq. 79 can be written in the following compact form:

$$y(t) = \frac{1}{a_2} \left[a_1 (y * u)(t) + a((y * u) * u)(t) + \ b_1 (x * u)(t) + b((x * u) * u)(t) \right]. \tag{80}$$

This diagram can be transfered to the diagram with only two integrators as shown in Fig. 61.

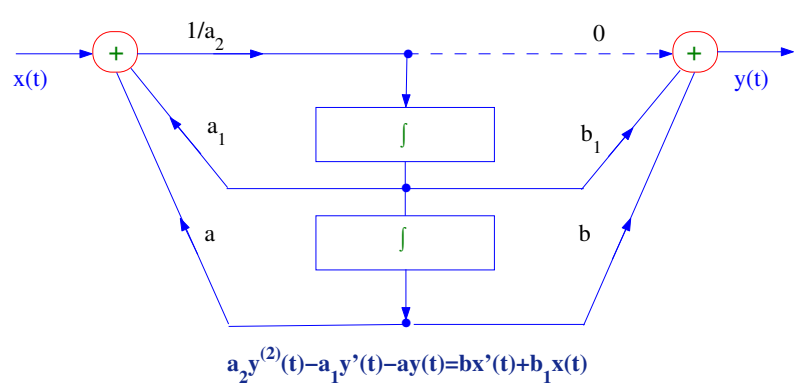


Fig. 61. Diagram of realization of the system in Form II.

This diagram of realization in Form II can be obtained from the diagram of Fig. 60 by replacing the left and right parts of the diagram as shown in Fig. 62

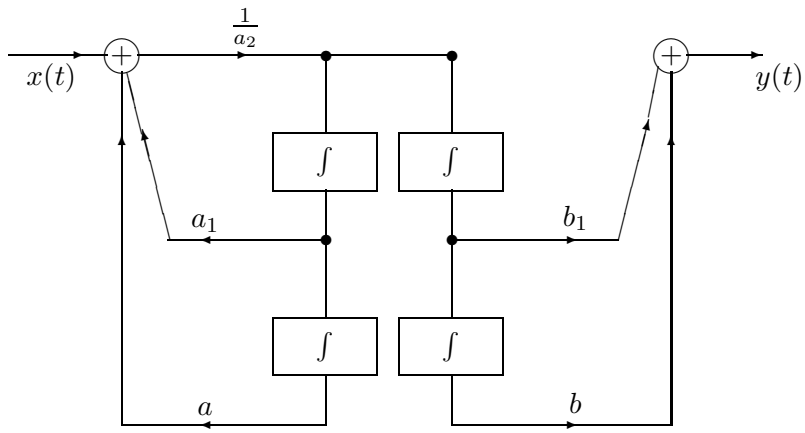


Fig. 62. (Step 1) Realization of the 2nd order LTI system of Eq. 78.

Then, since each pair of integrators have the same inputs and outputs, the diagram of Fig. 62 can be simplified as shown in Fig. 63.

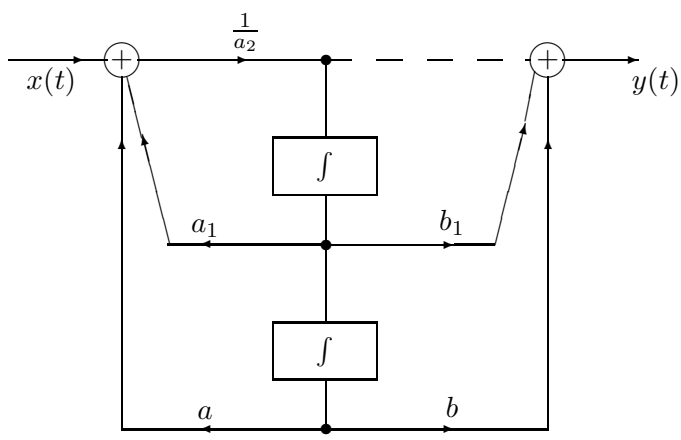


Fig. 63. (Step 2) Realization of the 2nd order LTI system of Eq. 78.

And this diagram is equals to the diagram of Fig. 61.

XII. The Fourier series: Periodic signals

In this section, we consider the representation of the time signals (functions) in another domain, the frequency domain. *Illustration:* Figures 65 and 66 show the signals (1-D and 2-D signals) and their representation in the frequency domain, which is called the Fourier series (or the discrete Fourier transform (DFT)). The representation in the frequency domain is unique.

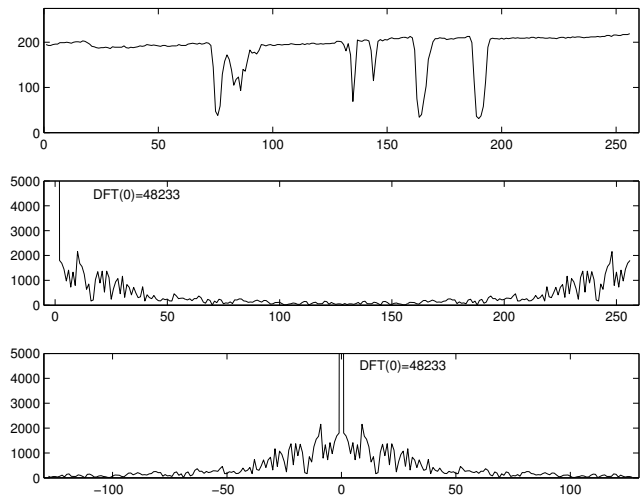


Fig. 64. (top) 1-D signal, (middle) coefficients of the Fourier series (FS), and (bottom) shifted FS to the middle. (FS is shown in the absolute scale)

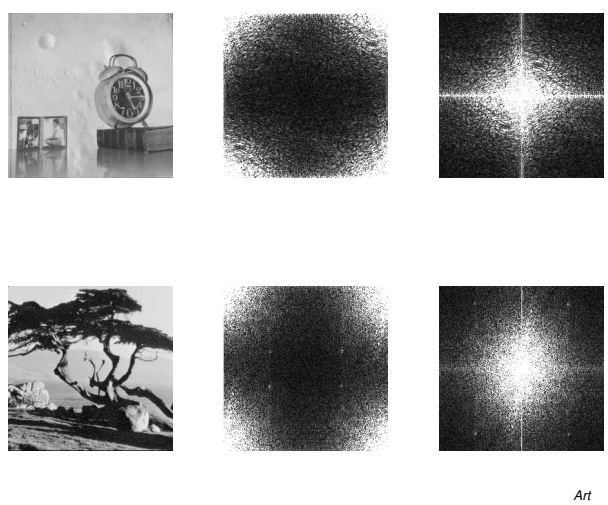


Fig. 65. (left) Two two-dimensional signals (images), (middle) coefficients of the Fourier series of images, and (right) shifted FSs to the middle. (FSs are shown in the absolute scale)

A. The concept of the Fourier series

We define the important concept of *the Fourier transform*, which is defined in a way similar to the transfer function $H(s)$ in the integral form of Eq. 50 (or Eq. 56), when the input function is the complex exponential function

$$x(t) = x_{js}(t) = e^{jst}, \qquad j^2 = -1,$$

which results in the transform function

$$H(js) = \int_{-\infty}^{+\infty} e^{-jst} h(t) dt. \tag{81}$$

Let us consider the following periodic signal

$$\begin{aligned} x(t) &= 10 + 3\cos(\omega_0 t) + 5\cos(2\omega_0 t - \pi/6) + 4\sin(3\omega_0 t) \\ &= 10 + 3 \cdot \cos(\omega_0 t) + 5A \cdot \cos(2\omega_0 t) + 0 \cdot \cos(3\omega_0 t) \\ &\quad + 0 \cdot \sin(\omega_0 t) + 5B \cdot \sin(2\omega_0 t) + 4 \cdot \sin(3\omega_0 t) \end{aligned}$$

where $A = \cos(\pi/6)$ and $B = \sin(\pi/6)$. The fundamental period of this signal is $T = 2\pi/\omega_0$. The illustration of the decomposition of this signal $x(t)$ is given in Figure 67.

Since, for each natural n,

$$\cos(n\omega_0 t) = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

$$\sin(n\omega_0 t) = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

we can write the above signal in the form

$$\begin{aligned} x(t) &= c_0 + c_1 e^{j\omega_0 t} + c_{-1} e^{-j\omega_0 t} \\ &\quad + c_2 e^{j2\omega_0 t} + c_{-2} e^{-2j\omega_0 t} \\ &\quad + c_3 e^{j3\omega_0 t} + c_{-3} e^{-3j\omega_0 t}, \end{aligned}$$

where the real or complex coefficients c_n, $n = 0, \pm 1, \pm 2, \pm 3$, can be easily defined.

Indeed,

$$\begin{aligned} x(t) &= 10 + 3\cos(\omega_0 t) + 5A\cos(2\omega_0 t) \\ &\quad + 5B\sin(2\omega_0 t) + 4\sin(3\omega_0 t) \\ &= 10 + 3\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + 5A\frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2} \\ &\quad + 5B\frac{e^{2j\omega_0 t} - e^{-j2\omega_0 t}}{2j} + 4\frac{e^{j3\omega_0 t} - e^{-j3\omega_0 t}}{2j} \\ &= 10 + \frac{3}{2}e^{j\omega_0 t} + \frac{3}{2}e^{-j\omega_0 t} \\ &\quad + \frac{5}{2}(A - jB)e^{j2\omega_0 t} + \frac{5}{2}(A + jB)e^{-j2\omega_0 t} \\ &\quad - 2je^{j3\omega_0 t} + 2je^{-j3\omega_0 t} \end{aligned}$$

and $c_0 = 10$, $c_1 = c_{-1} = 3/2$, $c_2 = 5/2(A - jB)$, $c_{-2} = 5/2(A + jB)$, and $c_3 = -2j$, $c_{-3} = 2j$.

So, the continuous-time periodic function $x(t)$ with period $T = 2\pi/\omega_0 = 2\pi$ is <u>completely</u> defined by seven coefficients c_n, i.e.,

$$\begin{aligned} x(t) &\leftrightarrow (c_0, c_1, c_{-1}, c_2, c_{-2}, c_3, c_{-3}) \\ &\leftrightarrow (10, 3/2, 3/2, 5/2(A - jB), 5/2(A + jB), -2j, 2j). \end{aligned}$$

This statement is very important: the function $x(t)$ defined (and/or periodic) in the interval $[0, 2\pi]$ is completely defined by seven coefficients.

Moreover, the following property of the complex conjugation takes place for the coefficients $c_{\pm n}$, $n = 1, 2, 3$,

$$c_{-1} = \bar{c}_1, \quad c_{-2} = \bar{c}_2, \quad c_{-3} = \bar{c}_3,$$

which means that it is enough to know only four coefficients to define the periodic signal $x(t)$ in the interval $[0, 2\pi]$,

$$\begin{aligned} x(t) &\leftrightarrow (c_0, c_1, c_2, c_3) \\ &\leftrightarrow (10, 3/2, 5/2(A - jB), -2j) \\ &\leftrightarrow (c_0, c_{-1}, c_{-2}, c_{-3}) \\ &\leftrightarrow (10, 3/2, 5/2(A + jB), 2j). \end{aligned}$$

Since $\omega = 1$ and $T = 2\pi$, we can write the signal $x(t)$ as follows:

$$\begin{aligned} x(t) &= 10 + 3\cos(t) + 5A\cos(2t) \\ &\quad +5B\sin(2t) + 4\sin(3t) \\ &= 10 + 3\cos(\frac{2\pi}{T}t) + 5A\cos(\frac{2\pi}{T}2t) \\ &\quad +5B\sin(\frac{2\pi}{T}2t) + 4\sin(\frac{2\pi}{T}3t) \\ &= c_0 + c_1 e^{j\frac{2\pi}{T}t} + c_{-1}e^{-j\frac{2\pi}{T}t} \\ &\quad + c_2 e^{j\frac{2\pi}{T}2t} + c_{-2}e^{-j\frac{2\pi}{T}2t} \\ &\quad + c_3 e^{j\frac{2\pi}{T}3t} + c_{-3}e^{-j\frac{2\pi}{T}3t}. \end{aligned}$$

The general result comes from 1800's and belongs to Jean Baptiste Joseph Fourier who has used for a periodic function $f(t)$ on an interval $[0, T]$ the expansion in the trigonometrical series (the *Fourier series*)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{p=1}^{+\infty} b_n \sin\frac{2\pi nt}{T} \tag{82}$$

for $t \in [0, T]$.

The coefficients a_n and b_n, $n = 1, 2, ...$, are calculated as

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi n}{T}t\right)\,dt \\ b_n &= \frac{2}{T}\int_0^T f(t)\sin(\frac{2\pi n}{T}t)\,dt. \end{aligned} \tag{83}$$

This Fourier series of the function $f(t)$ can also be written in the complex form

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\frac{2\pi t}{T}n}, \quad t \in [0, T], \tag{84}$$

with the coefficients

$$\begin{aligned} c_{\pm n} &= \frac{1}{T}\int_0^T f(t)e^{\mp j\frac{2\pi n}{T}t}\,dt = (a_n \mp jb_n)/2, \\ n &= 1, 2, \ldots, \quad (c_0 = a_0/2). \end{aligned} \tag{85}$$

Thus, the set of the coefficients c_n (or a_n, b_n) can be compared to the continuous function $f(t)$ and present the latter as a linear superposition of the harmonic oscillations (cosine and sine functions).

In the general $f(t)$ case, the function

$$(\mathcal{F} \circ f)(\omega) = F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{j\omega t}\,dt, \tag{86}$$

where $\omega \in (-\infty, +\infty)$, which is called *the Fourier transform* of the function f, can be compared to the absolute integrable function f, determined in the real line R^1,

$$\int_{-\infty}^{+\infty} |f(t)|dt < \infty.$$

The operation $\mathcal{F}$ is called an *1-D Fourier transformation.* The module of the function $|\mathcal{F} \circ f|$ is called an *energy Fourier spectrum* of the function f.

At that, the following inverse formula takes place:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{-j\omega t}\,d\omega, \quad t \in (-\infty, +\infty), \tag{87}$$

if the function f satisfies additional conditions (Dini), for instance, if f is continuous and has the finite derivative.

One can consider (86) as a transition from the time description of the function, f, to the description in the frequency domain, $F = (\mathcal{F} \circ f)$.

Figure 67 shows the decomposition of the signals

$$x(t) = 10 + 3\cos(\omega_0 t) + 5\cos(2\omega_0 t - \pi/6) + 4\sin(3\omega_0 t)$$

and

$$y(t) = 10 + 3\cos(\omega_0 t) + 5\cos(2\omega_0 t) + 4\cos(3\omega_0 t)$$

in the interval $[0, 3\pi]$. The phase $\phi = \pi/6)$ and the fundamental frequency $\omega_2 = 2$.

$$5\cos(2\omega_0 t - \pi/6) = 5\cos(2\omega_0 t)\cos(\pi/6) + 5\sin(2\omega_0 t)\sin(\pi/6)$$

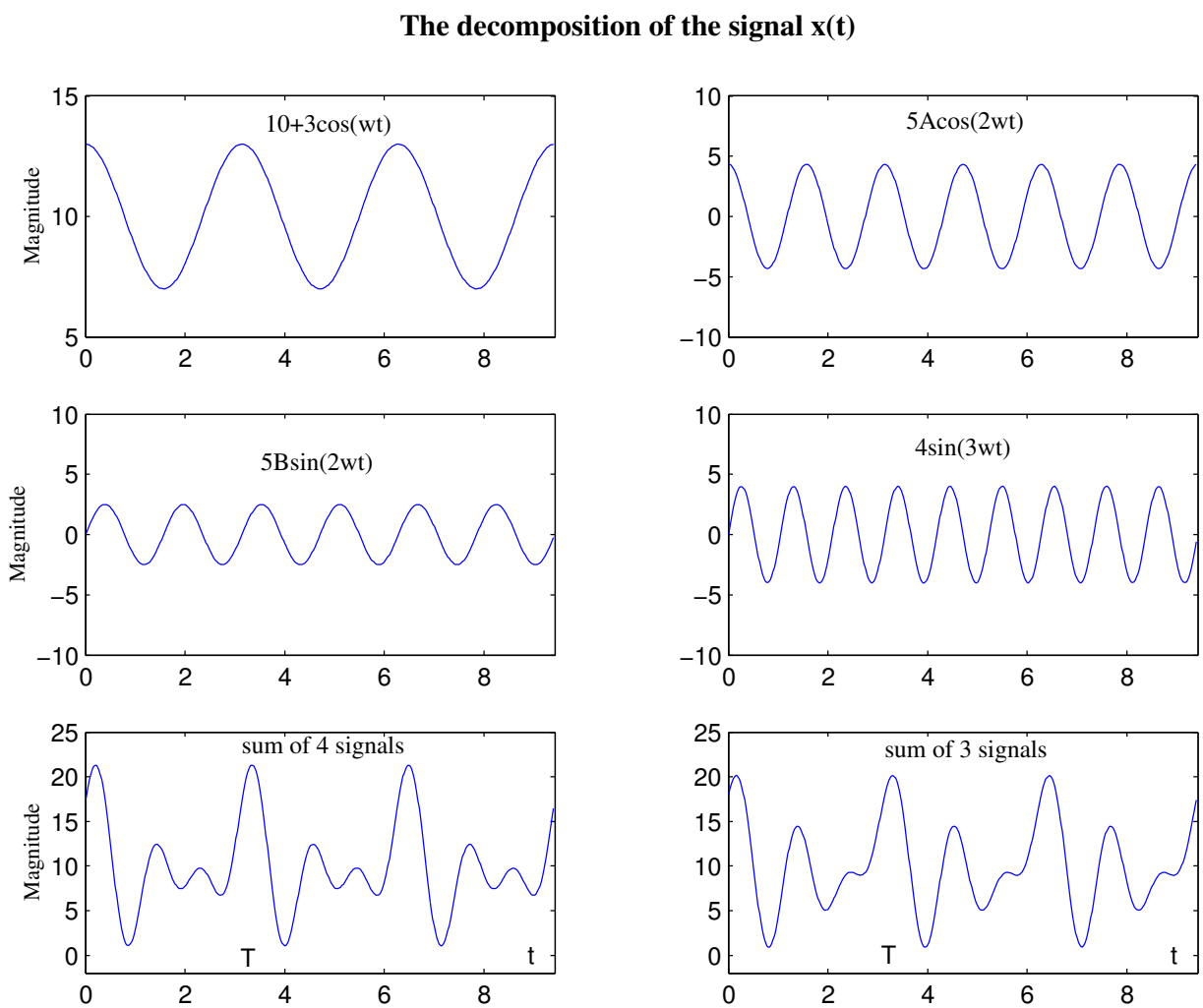


Fig. 66. Decomposition of the signal $x(t)$.

B. Main formulas for coefficients of Fourier Series

Given a periodic function $x(t)$ with period T can be represented in the three different ways which are *trigonometrical, exponential,* and *cosine* representation ($t \in [0, T]$):

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(n\omega t) + \sum_{n=1}^{+\infty} b_n \sin(n\omega t), \tag{88}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j(n\omega t)}, \tag{89}$$

$$x(t) = d_0 + \sum_{n=1}^{+\infty} d_n \cos(n\omega t + \vartheta_n), \tag{90}$$

where

$$\begin{aligned} &d_n = \sqrt{a_n^2 + b_n^2} = 2|c_n| = 2|c_{-n}|, \qquad n = 1, \ldots, \quad d_0 = a_0/2, \\ &c_n = \frac{a_n - jb_n}{2} \\ &c_{-n} = \frac{a_n + jb_n}{2}, \quad \text{and} \quad c_0 = a_0/2. \end{aligned} \tag{91}$$

Indeed, we can write the following in (88):

$$\begin{aligned} a_n \cos(n\omega t) + b_n \sin(n\omega t) &= \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos(n\omega t) + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin(n\omega t) \right) \\ &= \sqrt{a_n^2 + b_n^2} \left(\cos(\vartheta_n) \cos(n\omega t) + \sin(\vartheta_n) \sin(n\omega t) \right) \\ &= \sqrt{a_n^2 + b_n^2} \cos(n\omega t - \varphi_n), \end{aligned} \tag{92}$$

where

$$\cos(\varphi_n) = \frac{a_n}{\sqrt{a_n^2 + b_n^2}}, \quad \sin(\varphi_n) = \frac{b_n}{\sqrt{a_n^2 + b_n^2}}. \tag{93}$$

Therefore if $a_n \neq 0$, then the phase is calculated as follows:

$$\tan(\varphi_n) = b_n/a_n, \quad \varphi_n = -\vartheta_n = \tan^{-1}(b_n/a_n), \tag{94}$$

and $\varphi_n = \pi/2$ and $d_n = b_n$, if $a_n = 0$.

XIII. Basis Functions and Finite Fourier series

For a periodic function $f(t)$ defined on an interval $[0, T]$ the expansion in the trigonometrical series (the *Fourier series*) is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos \frac{2\pi nt}{T} + \sum_{n=1}^{+\infty} b_n \sin \frac{2\pi nt}{T}, \qquad t \in [0, T], \tag{95}$$

and this series of the function $f(t)$ is written in the complex form as

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\frac{2\pi t}{T}n}, \qquad t \in [0, T]. \tag{96}$$

The coefficients a_n and b_n, $n = 1, 2, ...$, and c_n, $n = 0, \pm 1, \pm 2, ...$, respectively are calculated by formulae (76) and (78). The set of the coefficients c_n (or a_n, b_n) can be compared to the continuous function $f(t)$. The function $f(t)$ is a linear superposition of the harmonic oscillations (cosine and sine functions), and coefficients a_n, b_n show the amplitudes of such oscillations.

Consider first the case of periodic functions $f(t)$ on the interval $[0, 2\pi]$, i.e., when the period $T = 2\pi$. We consider the linear space $L_2[0, 2\pi]$ of square-integrable functions on the interval $[0, 2\pi]$,

$$\int_0^{2\pi} |f(t)|^2 dt < \infty.$$

In the space $L_2[0, 2\pi]$, the set of functions

$$\begin{gathered} 1,\ \cos t,\ \cos 2t,\ \cos 3t, \dots \cos nt, \dots, \\ \sin t,\ \sin 2t,\ \sin 3t, \dots \sin nt, \dots \end{gathered} \tag{97}$$

compose a complete orthogonal system (*trigonometric* system) of functions.

In L_2 space, we define the inner product of two real functions $\phi_k(t)$ and $\phi_n(t)$ as

$$(\phi_k, \phi_n) = \int_0^{2\pi} \phi_k(t)\phi_n(t) dt$$

and the orthogonality means that $(\phi_1, \phi_2) = 0$.

The functions of (97) are orthogonal.

Indeed, if $k \neq n = 0$,

$$(\cos kt, 1) = \int_0^{2\pi} \cos kt\, dt = \int_0^{2\pi} \sin kt\, dt = 0.$$

If $k \neq n \neq 0$,

$$(\cos kt, \cos nt) = \int_0^{2\pi} \cos kt \cos nt\, dt = \frac{1}{2} \int_0^{2\pi} [\cos(k+n)t + \cos(k-n)t]\, dt = 0,$$

$$(\sin kt, \sin nt) = \int_0^{2\pi} \sin kt \sin nt\, dt = \frac{1}{2} \int_0^{2\pi} [\cos(k-n)t - \cos(k+n)t]\, dt = 0.$$

$$(\sin kt, \cos nt) = \int_0^{2\pi} \sin kt \cos nt\, dt = \frac{1}{2} \int_0^{2\pi} [\sin(k+n)t + \sin(k-n)t]\, dt = 0.$$

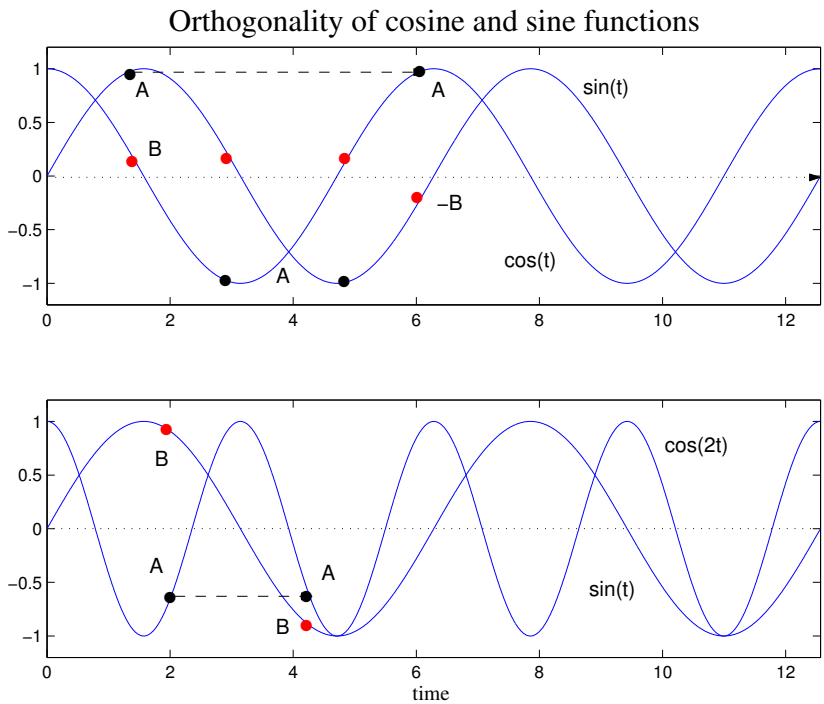


Fig. 67. Orthogonality of two pair of sinusoidal waves.

As an example, Figure 68 illustrates the orthogonality of functions $\cos(t)$ and $\sin(t)$ as well $\sin(t)$ and $\cos(2t)$.

If $k = n \neq 0$,

$$\begin{aligned}(\cos kt, \cos nt) &= \int_0^{2\pi} \cos^2 nt\, dt = \pi,\\ (\sin kt, \sin nt) &= \int_0^{2\pi} \sin^2 nt\, dt = \pi,\\ (1,1) &= \int_0^{2\pi} 1\, dt = 2\pi.\end{aligned}$$

Therefore, the set of functions

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \frac{\cos 3t}{\sqrt{\pi}}, \ldots,\\ &\qquad\quad \frac{\sin t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \frac{\sin 3t}{\sqrt{\pi}}, \ldots\end{aligned} \tag{98}$$

compose a complete orthonormal system of functions in $L_2[0, 2\pi]$.

Note, that when we write the Fourier expansion for the function $f \in L_2[0, 2\pi]$,

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} a_k \cos kt + \sum_{k=1}^{+\infty} b_k \sin kt, \qquad t \in [0, 2\pi]. \tag{99}$$

we understand that the *partial* sums, S_n converge to f,

$$S_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kt + \sum_{k=1}^{n} b_k \sin kt \to f(t), \ \forall t \in [0, 2\pi] \tag{100}$$

when $n \to +\infty$, and the convergence is in the metric of the space L_2, i.e.,

$$\int_0^{2\pi} |S_n(t) - f(t)|^2 \, dt \to 0$$

when $n \to +\infty$.

Moreover, for a given number n, the partial sum S_n gives the best approximation of the function f among all trigonometric polynomials,

$$T_n(t) = \frac{\alpha_0}{2} + \sum_{k=1}^{n} \alpha_k \cos kt + \sum_{k=1}^{n} \beta_k \sin kt.$$

Remark: System of functions in (97) compose the complete set of functions on the interval $[0, 2\pi]$. But, each of the following systems

$$1, \ \cos t, \ \cos 2t, \ \cos 3t, \dots \ \cos nt, \dots$$

and

$$\sin t, \ \sin 2t, \ \sin 3t, \dots \ \sin nt, \dots$$

compose the complete system of function only on the space $L_2[0, \pi]$, i.e., for functions defined (or periodic) on the interval $[0, \pi]$.

In the general case of the space $L_2[0, T]$ of square-integrable functions on the interval $[0, T$, we can use all mentioned above formulas changing 2π by T and basis functions $\cos nt$ and $\sin nt$ respectively by $\cos \frac{2\pi}{T} nt$ and $\sin \frac{2\pi}{T} nt$.

Denoting the frequency $\omega_0 = \frac{2\pi}{T}$, we obtain the complete orthogonal system of functions,

$$\begin{gathered} 1, \ \cos \omega_0 t, \ \cos 2\omega_0 t, \ \cos 3\omega_0 t, \dots \ \cos n\omega_0 t, \dots, \\ \sin \omega_0 t, \ \sin 2\omega_0 t, \ \sin 3\omega_0 t, \dots \ \sin n\omega_0 t, \dots \end{gathered} \tag{101}$$

in the space of functions $L_2[0, T]$.

Every function $f(t)$ of $L_2[0, T]$, can be described by its Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{+\infty} b_n \sin(n\omega_0 t), \qquad t \in [0, T]. \tag{102}$$

The coefficients a_n and b_n, $n = 1, 2, ...$, are calculated as

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt, \quad b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt. \tag{103}$$

In the comples form, the Fourier series of the function $f(t)$ is written as

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}, \qquad t \in [0, T], \tag{104}$$

with the coefficients

$$c_{\pm n} = \frac{1}{T}\int_0^T f(t)e^{\mp jn\omega_0 t}\,dt = (a_n \mp jb_n)/2, \quad n = 1, 2, \ldots, \quad (c_0 = a_0/2). \tag{105}$$

The Fourier series of Eq. 104 is defined for real and complex functions of the space $L_2[0, T]$. The inner product of two functions $\phi_k(t)$ and $\phi_n(t)$ of $L_2[0, T]$ is defined as

$$(\phi_k, \phi_n) = \int_0^T \phi_k(t)\bar{\phi}_n(t)dt.$$

The functions

$$\phi_n(t) = e^{jn\omega_0 t}, \quad n \in Z,$$

are orthogonal. Indeed, if $k \neq n$

$$\begin{aligned}
(\phi_k, \phi_n) &= \int_0^T e^{jk\omega_0 t}e^{-jn\omega_0 t}dt \\
&= \int_0^T e^{j(k-n)\omega_0 t}dt \\
&= \frac{1}{(k-n)\omega_0}e^{j(k-n)\omega_0 t}\Big|_0^T \\
&= \frac{1}{(k-n)\omega_0}\left[e^{j(k-n)\omega_0 T} - 1\right] = 0
\end{aligned}$$

because $\omega_0 T = 2\pi$.

And,

$$(\phi_n, \phi_n) = \int_0^T e^{jn\omega_0 t}e^{-jn\omega_0 t}dt = \int_0^T dt = T.$$

A. Finite Fourier series

Consider the Fourier series of the function $f(t)$ of the space $L_2[0,T]$,

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}, \qquad t \in [0,T], \tag{106}$$

or,

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{+\infty} b_k \sin(k\omega_0 t), \qquad t \in [0,T], \tag{107}$$

where $\omega_0 = 2\pi/T$.

Consider the discrete-time functions $f(t) = f(n)$, when $n = 0 : N-1$. In other words, we assume that the period T is an integer number, i.e., $T = N$. Then, we can write

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} a_k \cos(\frac{2\pi}{N}kt) + \sum_{k=1}^{+\infty} b_k \sin(\frac{2\pi}{N}kt), \tag{108}$$

which shows that the coefficients

$$a_k = b_k = 0, \qquad \forall k > N,$$

and (108) is of the form

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{N-1} a_k \cos(\frac{2\pi}{N}kt) + \sum_{k=1}^{N-1} b_k \sin(\frac{2\pi}{N}kt). \tag{109}$$

Indeed, the functions $\cos(\frac{2\pi}{N}kt)$ and $\sin(\frac{2\pi}{N}kt)$ are periodic with period N as the function of parameter k,

$$\cos(\frac{2\pi}{N}[k+N]t) = \cos(\frac{2\pi}{N}kt + 2\pi t) = \cos(\frac{2\pi}{N}kt), \quad t = 0 : N-1,$$

$$\sin(\frac{2\pi}{N}[k+N]t) = \sin(\frac{2\pi}{N}kt + 2\pi t) = \sin(\frac{2\pi}{N}kt), \quad t = 0 : N-1.$$

In the comples form, the finite Fourier series takes form

$$f(t) = \sum_{n=-N+1}^{N-1} c_n e^{j\frac{2\pi}{N}nt}, \qquad t = 0 : N-1. \tag{110}$$

where $c_0 = a_0/2$, and $c_n = (a_n - jb_n)/2$, $c_{-n} = (a_n - jb_n)/2$.

We can note, that the following property takes place for the complex coefficients $c_{\pm n}$:

$$c_{-n} = c_{N-n}, \qquad n = 1 : N-1.$$

Indeed,

$$e^{-j\frac{2\pi}{N}nt} = e^{-j\frac{2\pi}{N}nt + j\frac{2\pi}{N}Nt} = e^{j\frac{2\pi}{N}(N-n)t}$$

and Eq. 110 can be written as

$$f(t) = \sum_{n=0}^{N-1} c'_n e^{j\frac{2\pi}{N}nt}, \qquad t = 0 : N-1, \tag{111}$$

where

$$c'_n = c_{-n} + c_{N-n}, \qquad n = 1 : N-1.$$

In the case when the discrete-time function is real, the following take place

$$c'_n = \bar{c}'_{N-n}, \qquad n = 1 : N-1.$$

So to define the function $f(t)$ we can use $1 + 2N$ real coefficients

$$a_0, a_1, a_2, \ldots, a_{N-1}, b_1, b_2, \ldots, b_{N-1},$$

or $1 + N/2$ (or, $1 + [N+1]/2$) complex coefficients

$$c_0, c_1, c_2, \ldots, c_{N/2-1}, (c_{[N+1]/2-1}).$$

After changing the notation $c' \to c_p$, we define the finite Fourier series of the sequence $f(n)$ as

$$f(n) = \sum_{p=0}^{N-1} c_p e^{j\frac{2\pi}{N}pn}, \qquad n = 0 : N-1. \tag{112}$$

The coefficients c_p compose the *1-D N-point discrete Fourier transform* (1-D DFT) of the sequence $f(n)$, an can be defined as

$$c_p = F(p) = \frac{1}{N} \sum_{n=0}^{N-1} f(n) e^{-j\frac{2\pi}{N}np}, \qquad p = 0 : N-1. \tag{113}$$

and Eq. 112 describes the *inverse 1-D N-point discrete Fourier transform* (IDFT).

In MATLAB, we can use the following functions calculating the DFT:
1. *output=fft(input)* for the direct 1-D DFT of the sequence f
2. *output=ifft(input)* for the inverse 1-D DFT of the sequence f
3. *fftshift(input)* to swap the left and right halves of the 1-D DFT.

XIV. Exponential Fourier series representation

Given a periodic function $x(t)$ with period T, we consider the *exponential,* representation of $x(t)$, i.e., the *trigonometric Fourier series in complex form*:

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega t}, \quad t \in [0, T], \tag{117}$$

with the coefficients

$$\begin{aligned} c_n &= \frac{1}{T}\int_0^T x(t) e^{-j\frac{2\pi n}{T}t}\, dt, \quad n = \pm 1, \pm 2, \dots, \\ c_0 &= \frac{1}{T}\int_0^T x(t)\, dt, \qquad (\omega = 2\pi/T). \end{aligned} \tag{118}$$

Example 1: Let us consider the following periodic ramp function with period T,

$$x(t) = r\left(\frac{t}{T}\right) = \frac{t}{T}, \quad t \in [0, T].$$

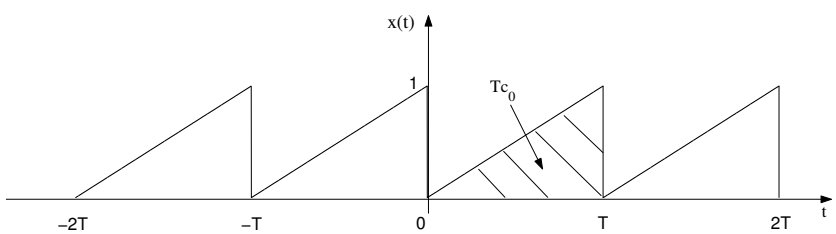


Fig. 72. Periodic ramp function.

The coefficients of the Fourier series of $x(t)$,

$$\begin{aligned} c_0 &= \frac{1}{T}\int_0^T r\left(\frac{t}{T}\right) dt = \int_0^T r\left(\frac{t}{T}\right) d\frac{t}{T} \\ &= \int_0^1 r(t)\, dt = \int_0^1 t\, dt = \frac{1}{2}. \end{aligned}$$

Further, by the direct calculations we obtain in the $n \neq 0$ case, that

$$\begin{aligned} c_n &= \int_0^T r\left(\frac{t}{T}\right) e^{-j\frac{2\pi n}{T}t}\, d\frac{t}{T} \\ &= \int_0^1 r(t)\, e^{-j2\pi nt}\, dt = -\frac{1}{j2\pi n}\int_0^1 t de^{-j2\pi nt} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{j2\pi n}\left[te^{-j2\pi nt}|_0^1 - \int_0^1 e^{-j2\pi nt}dt\right] \\
&= -\frac{1}{j2\pi n}\left[1 - 0 + \frac{1}{j2\pi n}\int_0^1 de^{-j2\pi nt}\right] \\
&= -\frac{1}{j2\pi n}\left[1 + \frac{1}{j2\pi n}e^{-j2\pi nt}|_0^1\right] \\
&= -\frac{1}{j2\pi n}\left[1 + \frac{1}{j2\pi n}[1-1]\right]
\end{aligned}
$$

we obtain

$$c_n = -\frac{1}{j2\pi n} = \frac{j}{2\pi n}, \quad n \neq 0.$$

Therefore,

$$x(t) = \frac{1}{2} + \sum_{n=\pm 1}^{+\infty} \frac{j}{2\pi n} e^{jn\omega t}, \quad t \in [0, T].$$

Owing to symmetry, $c_{-n} = -c_n$, for all integers $n > 0$, we have the following representation

$$
\begin{aligned}
x(t) &= \frac{1}{2} + \sum_{n=\pm 1}^{+\infty} \frac{j}{2\pi n} e^{jn\omega t} \\
&= \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{j}{2\pi n}\left[e^{jn\omega t} - e^{-jn\omega t}\right] \\
&= \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{j}{2\pi n} 2j\sin(n\omega t) \\
&= \frac{1}{2} - \sum_{n=1}^{+\infty} \frac{1}{\pi n}\sin(n\omega t). \qquad (119)
\end{aligned}
$$

Note, that the function $y(t) = x(t) - 0.5$ is odd, i.e., $y(-t) = -y(t)$, and $y(t)$ can be represented by the Fourier series of $\sin(\cdot)$ functions.

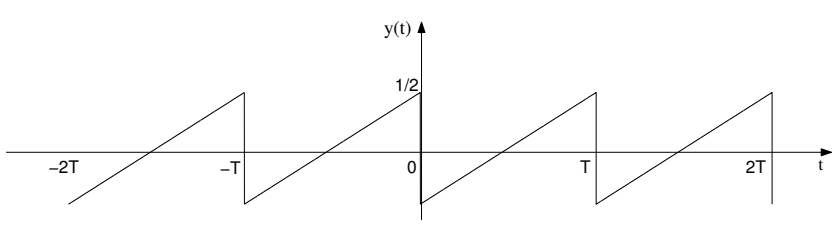


Fig. 73. Odd ramp function.

We can also write the cosine representation of $x(t)$,

$$x(t) = \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{1}{\pi n}\cos(n\omega t + \pi/2),$$

and the coefficients and phases of the cosine Fourier series

$$d_0 = \frac{1}{2}, \qquad d_n = \frac{1}{\pi n}, \quad \vartheta_n = \frac{\pi}{2}, \quad n = 1, 2, \dots .$$

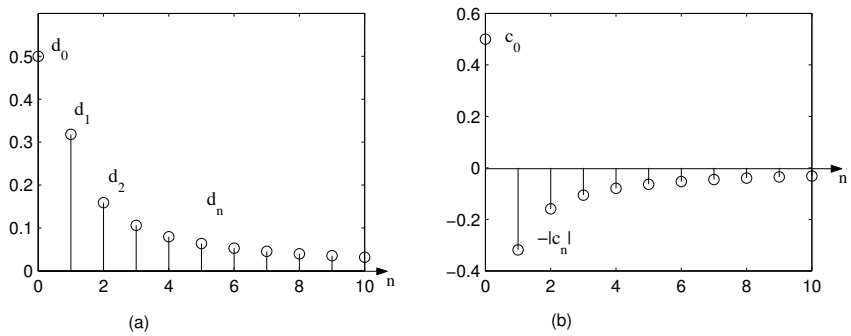


Fig. 74. Coefficients of the Fourier series.

Example 2: The similar representation has the following periodic step function, or "square-wave" function,

$$x_s(t) = \begin{cases} 1, & t \in [0, T/2), \\ -1, & t \in [T/2, T), \end{cases}$$

which shows the sign of the function $\sin(2\pi t/T)$.

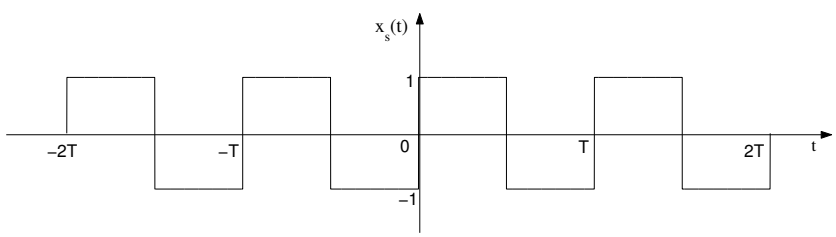


Fig. 75. Square-wave function.

Indeed, we first consider the Fourier series for the periodic pulse function

$$y(t) = x_s(t) + 1 = \begin{cases} 2, & t \in [0, T/2), \\ 0, & t \in [T/2, T). \end{cases}$$

Then,

$$c_0 = \frac{1}{T} \int_0^T y(t)\, dt = \frac{1}{T} \int_0^{T/2} 2\, dt = 1,$$

and for $n \neq 0$,

$$c_n = \frac{1}{T} \int_0^{T/2} 2e^{-j\omega n t}\, dt$$

$$
\begin{aligned}
&= -\frac{2}{T}\frac{1}{j\omega n}e^{-j\omega nt}\Big|_0^{T/2} \\
&= \frac{2}{T}\frac{1}{j\omega n}\left[1-e^{-j\omega Tn/2}\right] \\
&= \frac{1}{j\pi n}\left[1-e^{-j\pi n}\right], \qquad (\omega T = 2\pi)
\end{aligned}
$$

or,

$$
c_n = \begin{cases} \dfrac{2}{j\pi(2k+1)}, & n = 2k+1,\ k = 0, \pm 1, ... \\ 0, & n = 2k,\ k = 0, \pm 1, ... \ . \end{cases}
$$

Therefore,

$$
x_s(t) + 1 = 1 + \sum_{k=-\infty}^{+\infty} \frac{2}{j\pi(2k+1)} e^{j(2k+1)\omega t}, \quad t \in [0, T].
$$

and since $c_n = -c_{-n}$,

$$
\begin{aligned}
x_s(t) &= \sum_{k=-\infty}^{+\infty} \frac{2}{j\pi(2k+1)} e^{j(2k+1)\omega t} \\
&= \sum_{k=0}^{+\infty} \frac{2}{j\pi(2k+1)} \left[e^{j(2k+1)\omega t} - e^{-j(2k+1)\omega t}\right] \\
&= \sum_{k=0}^{+\infty} \frac{2}{j\pi(2k+1)} 2j \sin\left[(2k+1)\omega t\right] \\
&= \sum_{k=0}^{+\infty} \frac{4}{\pi(2k+1)} \sin\left[(2k+1)\omega t\right].
\end{aligned}
$$

We can compare the Fourier transforms of functions $x_s(t)$ and $x(t)$ of Example 1. For that, we take the function $2 - 4x(t)$, which can be written as (see Eq. 119)

$$
y(t) = 4\left[\frac{1}{2} - x(t) = \sum_{n=\pm 1}^{+\infty} \frac{1}{2\pi n} \sin(n\omega t)\right], \quad t \in [0, T].
$$

If we omit each second term from the Fourier series of $y(t)$, we obtain the Fourier series of the function $x_s(t)$.

Problem 1: (**a**). Find the coefficients of the periodic function $x(t)$ with period $2T$,

$$
x(t) = r\left(\frac{t}{T}\right) = \begin{cases} \dfrac{t}{T}, & t \in [0, T), \\ 0, & t \in [T, 2T). \end{cases}
$$

(**b**) Find the coefficients of the time-reversal version of the periodic function $x(t)$,

$$
x(-t) = r\left(-\frac{t}{T}\right) = \begin{cases} 0, & t \in [0, T), \\ 2 - \dfrac{t}{T}, & t \in [T, 2T). \end{cases}
$$

Remark. Note, that for a periodic function $x(t)$ with period T, we use to consider the intervals $[0,T]$ or $[-T/2,T/2]$. But, any other interval $[a,T+a]$ of length T can also be considered. Indeed, for all cases, the coefficients of the Fourier series c_n are the same. i.e.,

$$\begin{aligned} c_n &= \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi n}{T}t}\,dt, \quad n=0,\pm1,\pm2,\dots, \\ &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi n}{T}t}\,dt \qquad (\omega = 2\pi/T) \\ &= \frac{1}{T}\int_a^{a+T} x(t)e^{-j\frac{2\pi n}{T}t}\,dt, \quad a\in R. \end{aligned}$$

Indeed,

$$\begin{aligned} \int_a^{a+T} x(t)e^{-j\frac{2\pi n}{T}t}\,dt &= \left(\int_a^T + \int_T^{a+T}\right) x(t)e^{-j\frac{2\pi n}{T}t}\,dt \\ &= \int_a^T x(t)e^{-j\frac{2\pi n}{T}t}\,dt + \int_0^a x(t+T)e^{-j\frac{2\pi n}{T}(t+T)}\,dt \\ &= \int_a^T x(t)e^{-j\frac{2\pi n}{T}t}dt + \int_0^a x(t)e^{-j\frac{2\pi n}{T}t}\,dt \\ &= \int_0^T x(t)e^{-j\frac{2\pi n}{T}t}dt \end{aligned}$$

Example 3: We now consider the periodic function

$$x(t) = \begin{cases} \dfrac{t}{T}, & t\in[0,T), \\ 2-\dfrac{t}{T}, & t\in[T,2T), \end{cases}$$

for which

$$\begin{aligned} c_n &= \frac{1}{2}\int_0^T \frac{t}{T}e^{-j\frac{2\pi n}{2T}t}\,d\frac{t}{T} + \frac{1}{2}\int_T^{2T}\left[2-\frac{t}{T}\right]e^{-j\frac{2\pi n}{2T}t}\,d\frac{t}{T} \\ &= \frac{1}{2}\int_0^T \frac{t}{T}e^{-j\frac{2\pi n}{2T}t}\,d\frac{t}{T} + \frac{1}{2}\int_{-T}^0 -\frac{t}{T}e^{-j\frac{2\pi n}{2T}t}\,d\frac{t}{T} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 te^{-j\pi nt}\,dt + \frac{1}{2}\int_{-1}^0 -te^{-j\pi nt}\,dt \\
&= \frac{1}{2}\int_0^1 te^{-j\pi nt}\,dt + \frac{1}{2}\int_0^1 te^{j\pi nt}\,dt \\
&= \frac{1}{2}\int_0^1 t\left[e^{-j\pi nt} + e^{j\pi nt}\right]\,dt \\
&= \int_0^1 t\cos(\pi nt)\,dt = \frac{1}{\pi n}\int_0^1 td\sin(\pi nt) \\
&= \frac{1}{\pi n}\left[t\sin(\pi nt)|_0^1 - \int_0^1 \sin(\pi nt)dt\right] \\
&= \frac{1}{\pi n}\left[-\frac{1}{\pi n}\int_0^1 d\cos(\pi nt)\right] = -\left(\frac{1}{\pi n}\right)^2[1-\cos(\pi n)] \\
&= -\left(\frac{1}{\pi n}\right)^2[1-(-1)^n]
\end{aligned}
$$

So,

$$
c_n = -\left(\frac{1}{\pi n}\right)^2[1-(-1)^n] = c_n = \begin{cases} -\dfrac{2}{(\pi n)^2}, & n = 2k+1, \\ 0, & n = 2k. \end{cases}
$$

$c_0 = 1$, and the Fourier series of $x(t)$ is

$$
x(t) = 1 - \sum_{n=-\infty}^{+\infty}\left(\frac{1}{\pi n}\right)^2[1-(-1)^n]\,e^{jn\omega t}, \quad t \in [0,T].
$$

Since, $c_{-n} = c_n$, for all n, we can write

$$
x(t) = 1 - \sum_{n=\pm1,\pm2,\ldots}\left(\frac{1}{\pi n}\right)^2[1-(-1)^n]\,2\cos(n\omega t),
$$

or

$$
x(t) = 1 - \sum_{k=0}^{+\infty}\frac{4}{[\pi(2k+1)]^2}\cos((2k+1)\omega t).
$$

XV. The Fourier series and FFT with MATLAB

In this section, we consider the simple script of the program to calculate the Fourier transform of a periodic signal $x(t) = x(n)$ with integer time points. The input signal is considered to be the following:

$$x(t) = 1 + 2\cos(2\pi f_1 t) + 4\sin(2\pi f_2 t), \quad f_1 = 0.02(\text{Hz}),\ f_2 = 0.03(\text{Hz}). \tag{120}$$

The signal is periodic with the fundamental frequency $f_0 = 0.01$Hz. where $\omega_0 = 2\pi \cdot 0.01$ (in rad/s). The carrier frequencies f_1 and f_2 are integer multiple to f_0, i.e., $f_1 = 2f_0$ and $f_2 = 3f_0$. Therefore, the fundamental period and frequency ω_0 (in rad/s) of this signal are

$$T = 1/f_0 = 100 \text{ s}, \quad \omega_0 = 2\pi f_0 = 2\pi \cdot 0.01 \text{ rad/s} = 0.02\pi \text{ rad/s}.$$

This signal $x(t)$ in the interval $[0, 300]$ is given in Figure 76. The representation of the first period (we also call it *the fundamental period*), which we denote by $x_1(t)$, in the frequency domain in the absolute scale also is shown in the figure. This representation is also called the 100-point discrete Fourier transform (DFT) of the signal $x_1(t)$.

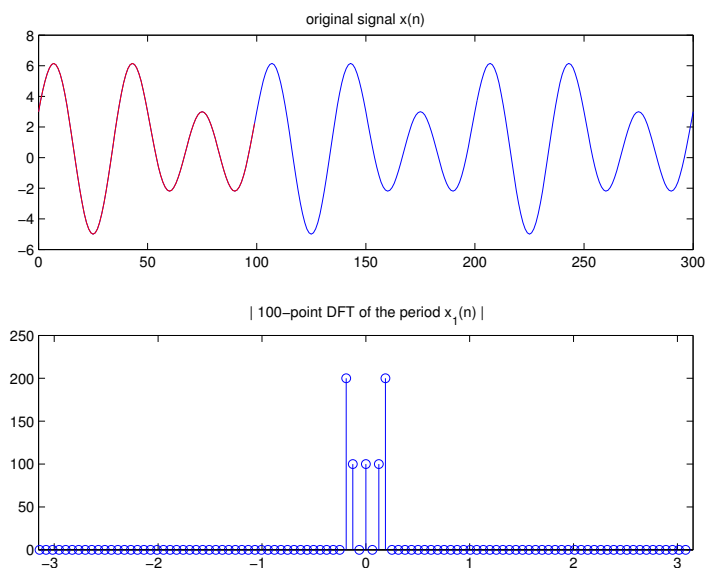


Fig. 76. (top) 1-D signal $x(t)$ and (bottom) the shifted 100-point DFT of the first period $x_1(t)$. (DFT is shown in the absolute scale)

Below are the scripts of the program which was written in the class to calculate the signal $x(t)$ and its 100-point DFT. This script (maybe with some small changes) was also posted in the web cite (BB).

```
% ==========================================================================
% call: work_withFS.m (the first draft)
%
% for EE 3424 (Math in SS), ECE UTSA
% Artyom Grigoryan (from the class work on October 26, 2015)
```

```
% 1. Signal composition
% x(t) = 1 + 2cos(2pi0.02t) + 4sin(2pi0.03t)

t_end=300;
t=0:t_end;

f0=0.01;

f1=2*f0; % in Hz
f2=3*f0;
w1=2*pi*f1;  w2=2*pi*f2;

N=length(t);
x=zeros(1,N);
x = 1 + 2*cos(w1*t) +4*sin(w2*t);

% 2. Display the signal
figure;
subplot(2,1,1);
plot(t,x);
title('original signal x(n)');

% 3. Take the fund. period of the signal
T=1/f0;
x1=zeros(1,T);
x1=x(1:T);
t1=t(1:T);

hold on;
plot(t1,x1,'r');

% 4. Calculate the FS (X1) of the signal x1
X1=fft(x1);
X1abs=abs(X1);
w=linspace(0,2*pi-2*pi/T,T);

% 5. Plot the DFT of the signal
subplot(2,1,2);
stem(w,X1abs);
title('| Fourier series of x1 |');
axis([ 0,2*pi+0.01, 0,250]);

pause(2)
axis([ 0,2*pi/5+0.01, 0,250]);

pause(2)
X1abs_shifted=fftshift(X1abs);

w1=w-pi;
stem(w1,X1abs_shifted);
title('| 100-point DFT of the period x_1(n) |');
axis([ -pi-0.01,pi+0.01, 0,250]);
% ==========================================================================================
```

A. Full Script of the Program with Description and Explanation

Now, I will try to explain the above given script and correct it. Namely, we will see the relationship between the 100-point DFT of the period $x_1(t)$ and the coefficients C_k of the Fourier series of the signal $x(t)$.

1. Note, the periodic signal $x(t)$ is given in the integer interval $[0, 300]$ and the coefficients C_k should be calculated for this signal. The 100-point DFT is calculated on the first period $x_1(t)$ of the signal, not on the entire signal $x(t)$.

2. The period of the signal $N = T = 100$ is integer. The fundamental frequency $\omega_0 = 2\pi/N$ (in rad/s).

3. The finite Fourier series of the sequence $x(n)$ is

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j\frac{2\pi}{N}kn}, \qquad n = 0, 1, 2, 3, \ldots. \tag{121}$$

The coefficients C_k are calculated as

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)e^{-j\frac{2\pi}{N}nk}, \qquad k = 0 : N-1. \tag{122}$$

4. The N-point discrete Fourier transform of the sequence $x_1(n)$ is calculated by "fft.m" function in MATLAB as

$$X_k = \sum_{n=0}^{N-1} x_1(n)e^{-j\frac{2\pi}{N}nk}, \qquad k = 0 : N-1. \tag{123}$$

<u>Therefore</u>, to calculate the coefficients c_k from the values of F_k, the following normalization of the N-point DFT should be considered:

$$C_k = \frac{1}{N}X_k, \qquad k = 0 : N-1.$$

Now, we consider the script of the program in more detail. The script of the program "work_withFS1.m" is given below; it uses the functions which are in the standard library of MATLAB:

1. "fft.m" – for the direct 1-D DFT of the signal
2. "fftshift.m" to swap the left and right halves of the 1-D DFT,
3. functions to display and draw graphics, such as "figure, plot, stem, text, xlabel, ..." One can use the command "help" to know more about these functions. For instance, for "stem" on can write in the command window of MATLAB the command

>> help stem

Comments: When start MATLAB, use the command "cd" which means "change directory" to move away from the folder "MATLAB" and work in another folder, as I do in the class when writing in the command line the following:

>> cd C:/Art/Work_inclass

The program "work_withFS1.m" calculates the signal, its coefficients of the Fourier series, and prints the results as shown in Figure 77.

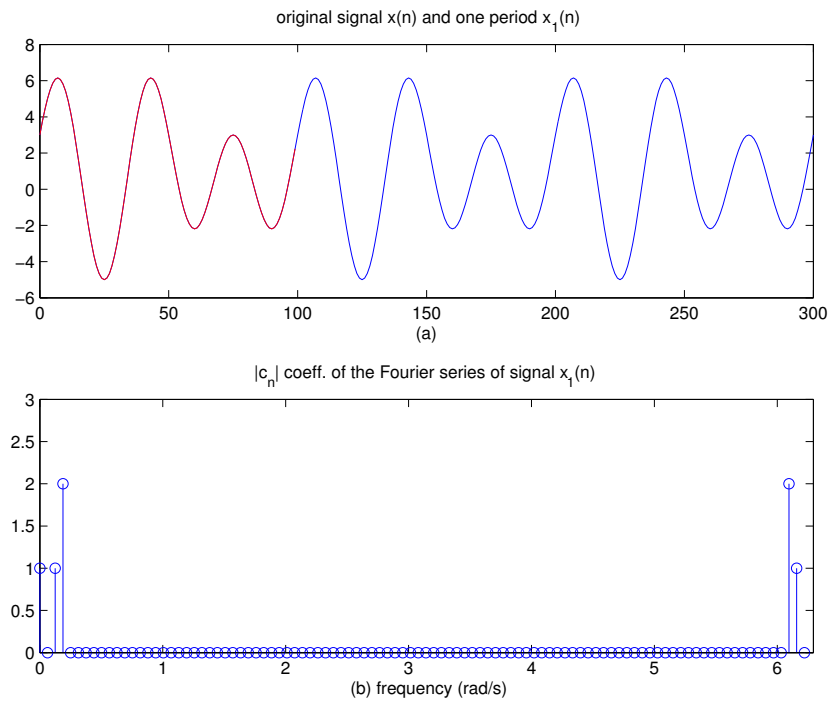


Fig. 77. (top) 1-D signal $x(t)$ and (bottom) the shifted 100-point DFT of the first period $x_1(t)$. (DFT is shown in the absolute scale)

Code Description:

```
% ========================= comments ========================================
% call: work_withFS1.m                  (from class work of October 26, 2015)
% ---------------------------------------------------------------------------
% To demonstrate in detail the example of script written for
% composing the signal and calculating its Fourier series coefficients Cn
% by using the fast Fourier transform (FFT) function fft.m from MATLAB.
%
% For class EE 3424 Math in Signals and Systems
% Artyom Grigoryan, October 26-27, 2015, ECE Dept. UTSA
% ---------------------------------------------------------------------------

% ---------------------------------- PART 1 -------------------------------------
% 1. Signal composition
% x(t) = 1 + 2cos(2pi0.02t) + 4sin(2pi0.03t),        [pi stands for 3.1416...]
% in the integer interval [0,300], or [0,t_end]

t_end=300;
t=0:t_end;                % the vector of time points 0,1,2,...,300
f0=0.01;                  % the fundamental frequency of the signal x(t) in Hz
f1=2*f0; f2=3*f0;         % two frequencies of the signal x(t) in Hz
w1=2*pi*f1;  w2=2*pi*f2; % two frequencies of the signal x(t) in rad/s
```

```
%
N=length(t);             % the length of the signal x(t), which is N=301
x=zeros(1,N);            % asking memory for the new variable (signal x(t))
x = 1 + 2*cos(w1*t) +4*sin(w2*t);    % calculation of the signal at all time points
%
% 2. Display the signal
figure;                  % open the window for the figure
subplot(2,1,1);          % the window is divided by 2 parts and 1st part will be used
plot(t,x);               % plot the signal x(t) versus the time points t
title('original signal x(n) with 3 periods');   % add the title to the graph
xlabel(' (a) ');         % label the x-axis as "(a)"

% 2. Calculate select the fundamental period of the signal
T=1/f0;                  % it is 100 for this example
% 3. Calculate and plot the first period of the signal, which is denoted as x1(t)
x1=zeros(1,T);           % asking memory for x1(t); T zeros will be filled in x1
x1=x(1:T);               % assign the first T values of the signal x(t) to x1(t)
t1=t(1:T);               % time points for the period x1(t)

pause(2)                 % pause for 2 seconds (it is good to use when running the code)
% draw or highlight the period x1(t) by the red color on the original signal
hold on;                 % hold the graph of x(t)
plot(t1,x1,'r');         % plot in red the signal x1(t) versus the time points t1
title('original signal x(n) and one period x_1(n)');   % change the title of the graphs
% The result of code on this stage is shown above in Figure 77 (top)

% --------------------------------- PART 2 -------------------------------------------
% 1. Calculate the Fourier series (FS) of the signal x(t) by using the function fft

X1=fft(x1);              % or fft(x1,T);  calculate the T-point DFT of the signal x1(t)
                         % according to equation (111). Here X1 stands for F.
X1abs=abs(X1);           % calculate the DFT in the absolute scale; for plotting only
%   X1real=real(X1);     % calculate the real part of the DFT
%   X1imag=imag(X1);     % calculate the imaginary part of the DFT
%   X1ang=phase(X1);     % calculate the phase part of the DFT [angle(X1) can be used]

% DFT and FS will be plotted versus the T frequency-points in the interval [0,2*pi).
w=linspace(0,2*pi-2*pi/T,T);    % T or 100 frequency-points are written in the vector w

% (!!) Consider the relationship between the FFT and Cn coefficients:
cn_coefficients=X1/T;                          % calculation of Cn coefficients
cn_coefficients_inabs=abs(cn_coefficients);    % Cn in the absolute scale

% The following string variable will be used for the title of FS
s_title='|c_n| coeff. of the Fourier series of signal x_1(n) ';

subplot(2,1,2);          % the 2nd sub-window will be used for the linear spectrum |FS|
stem(w,cn_coefficients_inabs);    % plot coefficients Cn versus the frequency-points
                                  % one can use plot function as well instead of "stem"
title(s_title);                   % title will be added to the graph
axis([ 0,2*pi+0.01, 0,3]);        % to correct a little the window for the graphic
xlabel(' (b) frequency (rad/s)'); % add the label along x-axis
% --------------  The result of the code on this stage is shown in Figure 78. ------------
```

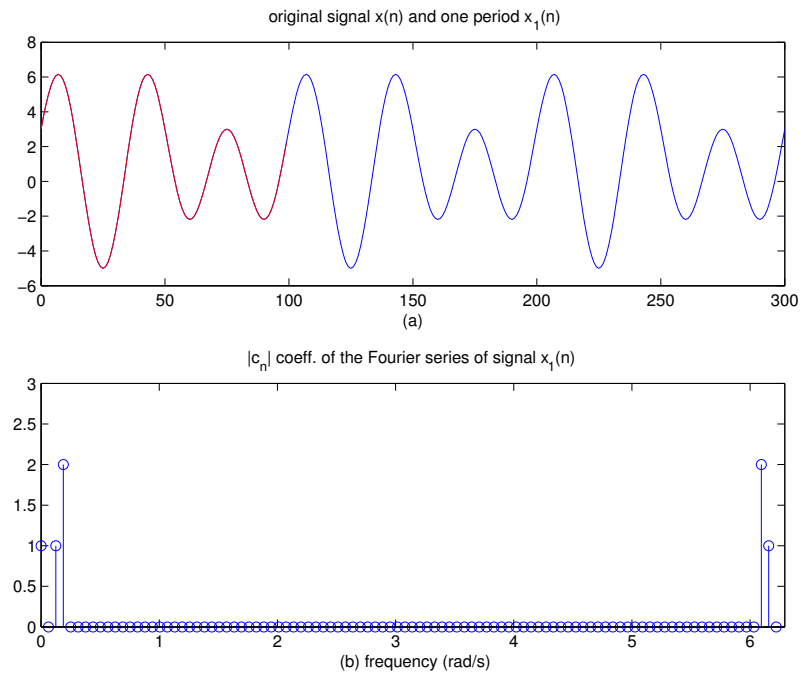


Fig. 78. The result of the code.

The same linear spectrum can be plotted versus the frequency-points in Hz with the following similar commands. The frequency points will be taken from the interval $[0, 1]$ as the results of division of the ω-frequency interval $[0, 2\pi]$ by 2π, since $f = \omega/(2\pi)$.

```
figure;                              % open the new figure  (Figure 79)
subplot(2,1,2);                      % the 2nd sub-window will be used for |FS|
f=linspace(0,1-1/T,T);               % or f=w/(2*pi);  frequency-points in Hz
stem(f,cn_coefficients_inabs);       % plot coefficients Cn versus the frequency-points
axis([ 0-0.02,1+0.02, 0,3]);         % to correct the window for the graphic
title(s_title);                      % add the title to the graph
xlabel(' (b) frequency (Hz)');       % add the label along x-axis
pause(2)                             % use this line when adding this part to the code
```

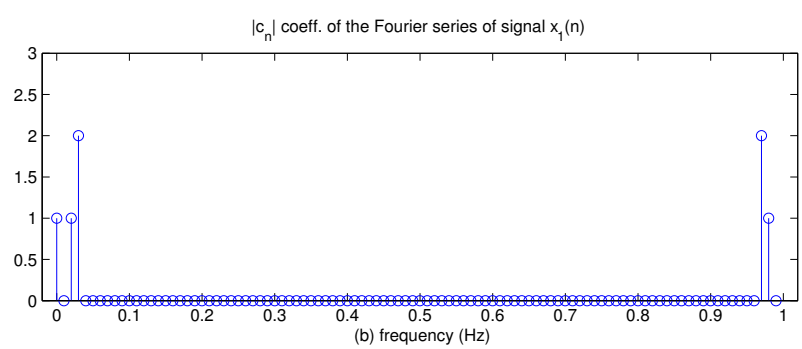


Fig. 79. The C_n coefficients versus the frequency-points in Hz.

The same linear spectrum can shown only in the small sub-interval of the frequency-points.

```
% -------------------------- Addition commands ------------------------------------------

figure;                              % open the new figure
subplot(2,1,2);                      % the 2nd sub-window will be used for |FS|
h_st=stem(w,cn_coefficients_inabs); % plot coefficients Cn versus the frequency-points

% by using the pointer h_st, one can change the properties of the graph as shown here:
set(h_st, 'MarkerSize',4, ...
          'MarkerEdgeColor','b', ...
          'MarkerFaceColor','g');

axis([ 0-0.02,2*pi/5+0.02, 0,3]);   % to correct the window for the graphic
title(s_title);                      % add the title to the graph
xlabel(' \omega  frequency (rad/s)');     % add the label along x-axis

pause(2)                              % use this line when adding this part to the code

% --------------  The result of the code on this stage is shown in Figure 80. ------------
% ---------------------------------------------------------------------------------------
```

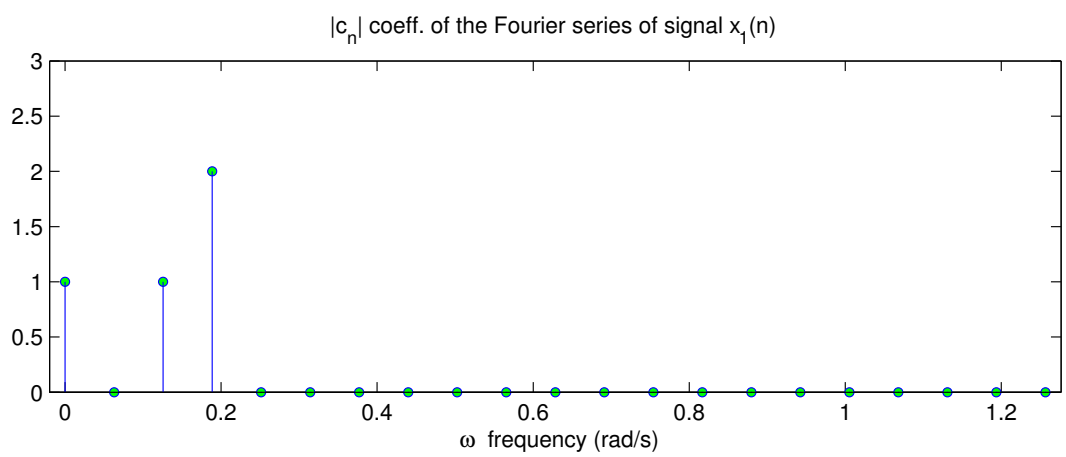


Fig. 80. Additional commands: The first 21 C_n coefficients versus the frequency-points in rad/s.

The linear spectrum can be plotted by shifting or cyclicly shifting the coefficients to the middle, as shown in Figure 81.

```
% -------------------------- Addition commands ------------------------------------------

figure;                      % open the new figure
subplot(2,1,2);              % the 2nd sub-window will be used for |FS|

% use command fftshift to shift Cn coefficients cyclically the graph to the middle point
cn_coefficients_inabsandshifted=fftshift(cn_coefficients_inabs);

% These coefficients should be plotted versus new frequency-points, namely points in the
% interval [-pi,pi]. A simple way is to use the interval for frequency-points w, as shown
```

```
w1=w-pi;

h_st2=stem(w1,cn_coefficients_inabsandshifted);  % plot shifted coefficients Cn

% Here, we change the properties of the graph as
set(h_st2,'MarkerSize', 4, ...
    'MarkerEdgeColor','b', ...
    'MarkerFaceColor','b');

title(s_title);                                  % add the title to the graph
axis([ -pi-0.05,pi+0.05, 0,3]);                  % to correct the window for the graphic
xlabel(' \omega  frequency (rad/s)');            % add the label along x-axis

text(-2.75,2.5,' |c_{-n}|=|c_{100-n|   (T=100)') % add text at points (-2.75,2.5)

% We can save this figure as the eps-file with the name "figure_fs5.eps" as
  print -depsc figure_fs5.eps;

% --------------  The result of the code on this stage is shown in Figure 81. ------------
% ------------------------------------------------------------------------------------------
```

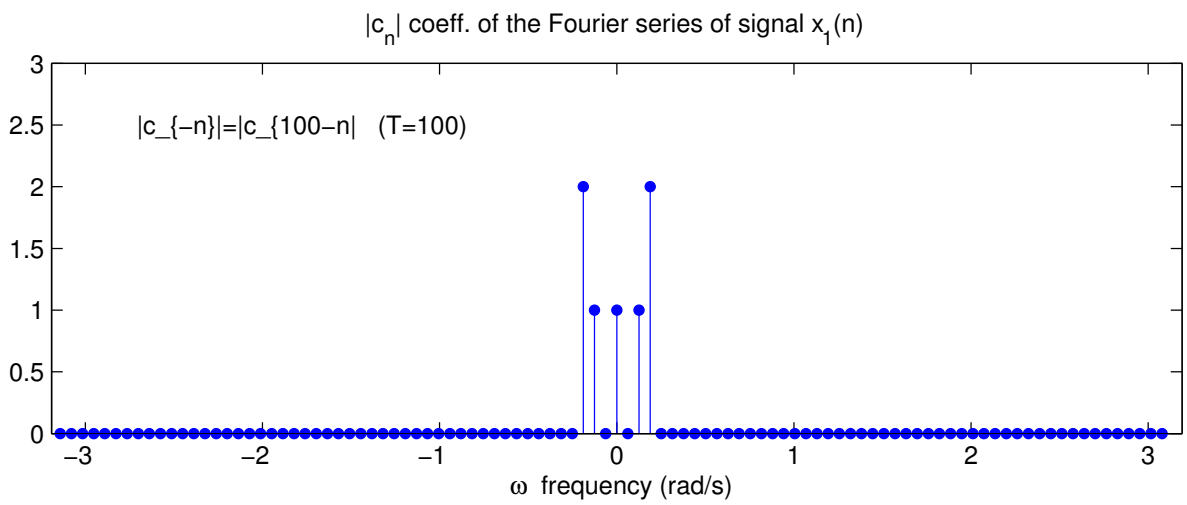


Fig. 81. The linear spectrum of the signal $x(t)$, or C_n coefficients versus the frequency-points in rad/s.

```
% ------------------------------------------------------------------------------------------
```

I hope you will run and analyze all lines of these scripts and it will help you in writing your own codes in DSP.

Instructor: Dr. Artyom Grigoryan
10/28/2015

XVI. The Fourier series application: Ideal low pass filter with MATLAB

In this section, we consider the simple application of the concept of the Fourier series, or discrete Fourier transform in de-noising of signal, by illuminating the coefficients of the Fourier series.

Ideal Low-Pass Filter: Assume, that the coefficients of the Fourier series of the periodic signal $x(t)$ are plotted versus the frequencies, i.e., consider

$$C_n = C(\omega_n), \quad n = 0, 1, 2, ..., (N-1),$$

where

$$\omega_n = n\omega_0, \quad n = 0, 1, 2, ..., (N-1).$$

Here, $\omega_0 = 2\pi/N$ is the fundamental frequency (in rad/s) of the signal, and the integer N its period. Given a frequency $\omega_{cutoff} = \omega_{n_0}$, the frequencies

$$0, \omega_1, \omega_2, \cdots, \omega_{n_0-1}, \omega_{n_0}$$

are considered are low frequencies together with the frequencies

$$\omega_{n_0+1}, \omega_{n_0+2}, \cdots, \omega_{N-2}, \omega_{N-1},$$

since the coefficients of the Fourier series are periodic, i.e.,

$$C_{N-n} = C_{-n}, \quad \text{or} \quad C(2\pi - \omega_n) = C(-\omega_n).$$

Figure 82 shows such a low-pass filter (LPF) with the cut-off frequency

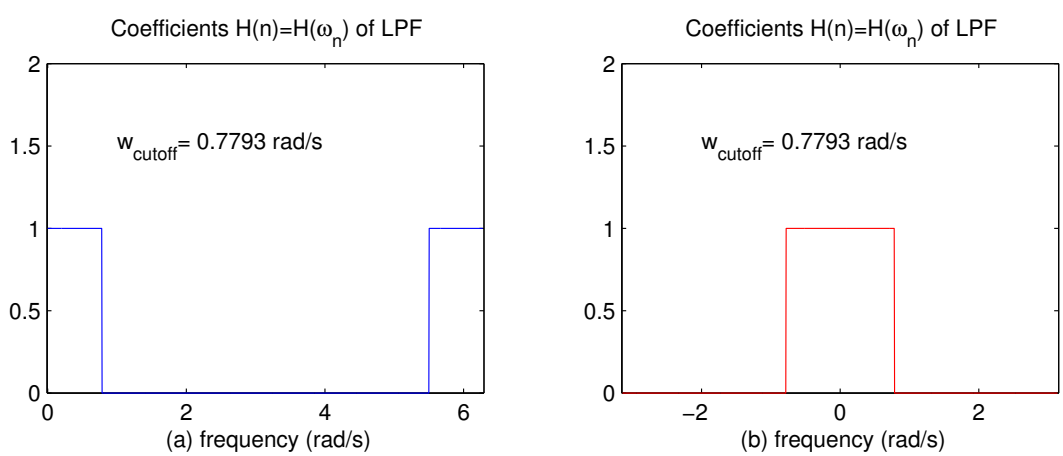


Fig. 82. Frequency characteristic of the LPF (a) before and (b) after the shifting.

$$\omega_{cutoff} = \omega_{127} = 2\pi/N \cdot 127 = 0.7988 \text{ rad/s}.$$

We denote the coefficients of this LPF by $H_0, H_1, H_2, \ldots, H_{N-2}, H_{N-1}$.

The operation of signal filtration by this filter is described as

$$C_n \to \hat{C}_n = H(n)C_n, \quad n = 0, 1, 2, ..., (N-1).$$

A new signal with the Fourier series coefficients $\hat{C}_n$ is the filtered signal $\hat{x}$, i.e.,

$$\hat{x}(n) = \sum_{k=0}^{N-1} \hat{C}_k e^{j\frac{2\pi}{N}kn}, \qquad n = 0, 1, 2, 3, \ldots (N-1). \tag{124}$$

Below is the scripts of the program which was written in the class to calculate the operation of filtration signal

$$x(n) \to \hat{x}(n)$$

over the signal of length $N = 512$. This script (maybe with some small changes) will also be posted in the web cite (BB).

```
% ============================ comments ========================================
% call: work_lpf_inMSS.m                (from class work of October 28, 2015)
% ---------------------------------------------------------------------------------
% Example of application of the Fourier series (or discrete Fourier transform):
% Ideal filter in the frequency domain with a given cut-off frequency
% The coefficients of this filter take values 1 and 0, namely
%    1; in the region of low-frequencies (around zero)
%    0; outside this region
% This filter is used in the code to filter the noise on the signal
%
% For class EE 3424 Math in Signals and Systems
% Artyom Grigoryan, October 26-27, 2015, ECE Dept. UTSA
% ---------------------------------------------------------------------------------

   clear all; close all;      % clear all data used before and close all figures
% ------------------------------------- part 1 ---------------------------------------
% 1. Signal composition from the binary file
   S1_name='data2.8';                       % file name with the signal data
   fid=fopen(S1_name,'rb');                 % open to r(read) b(inary) file with name
   X=fread(fid);  fclose(fid); clear fid;   % read the file, close it and remove fid
   N=length(X);                             % the length of this signal is large
   N=512;                                   % only the first 512 values will be used
   x=X(1:N);   clear X;                     % x(n) is the new signal, clear X-data

   figure;                          % open the window for the new figure
   subplot(2,2,1);                  % the window is divided by 2x2 parts, use the 1st
   plot(x);                         % plot x(n) versus the time points n=1,2,3,..., N
   axis([0,512,0,40]);              % to correct a little the window for the graph
   S_title='Original signal x(1:)';  % title for the graph
   h_title=title(S_title);          % add the title to the graph
   set(h_title,'Color','b','FontName','Times','FontSize',10);    % change its properties
   xlabel(' (a) ');                 % label the x-axis as "(a)"
   pause(2)                         % use this line when running the code

% 2. 512-point DFT calculation (which is the Fourier series up to the constant N)
   X=zeros(1,N);                    % memory for the new variable for the DFT, X
```

```
X=fft(x);  % or fft(x,N);          % calculation of the 512-point FDT of x(n)
Xabs=fftshift(abs(X));             % cyclically shift of the magnitude of the DFT

subplot(2,2,2);                    % the 2nd sub-window will be used for the |DFT|
plot(fftshift(Xabs));              % plot the original (not shifted) DFT in abs. scale
axis([0,512,0,1000]);              % to correct a little the window for the graph
title('512-point |DFT| and Ideal LPF');            % add the title to the graph
xlabel(' (b) ');                   % label the x-axis as "(b)"
pause(2)                           % use this line when running the code
```

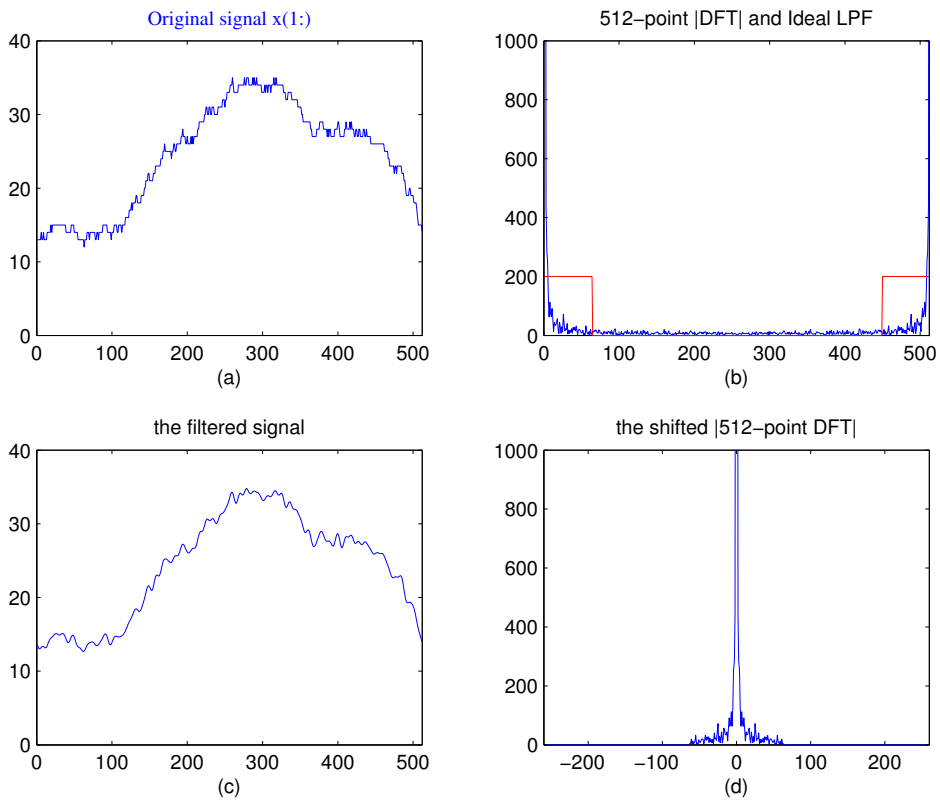


Fig. 83. (a) The original signal, (b) the 512-point DFT in absolute scale and the LPF, (c) the result of filtration, and (d) the filtered spectrum of the signal.

```
% ---------------------------------------------------------------------------------------
%3.  calculate the low-pass filter (LPF)
   wcut=64;                   % cut-off point (integer, not frequency) for the LPF
   H=ones(N,1);               % memory for the new variable with all 1s for the DFT, H
   H(wcut+1:N-wcut+1)=0;      % fill hight frequencies with zero values
   hold on;                   % hold on the current active part - subplot(2,2,2)
   plot(H*200,'r');           % plot the LP Filter after amplifying by the factor of 400
   pause(2)                   % use this line when running the code

%4. Filtration
   X1=H.*X;                          % the operation is reduced to the multiplication
   subplot(2,2,4);                   % the 4th sub-window will be used for the |X1|
```

```
   w1=-256:255;                        % new frequency-points for the shifted spectrum
   plot(w1,fftshift(abs(X1)));         % plot the result of multiplication of spectra
   axis([-260,260,0,1000]);            % to correct a little the window for the graph
   title('the original and filtered signals');       % add the title to the graph
   xlabel(' (d) ');                                  % label the x-axis as "(d)"
   pause(2)                            % use this line when running the code

%4. Calculate and plot the result of filtration of high frequencies
   subplot(2,2,3);                     % the 3rd sub-window will be used for signal
   x1=real(ifft(X1));                  % the filtered signal x1(n)
   plot(real(ifft(X1)));               % plot the signal x1(n)
   axis([0,512,0,40]);                 % to correct a little the window for the graph
   title('the filtered signal');       % add the title to the graph
   xlabel(' (c) ');                    % label the x-axis as "(c)"
   pause(2)                            % use this line when running the code
% --------------  The result of the code on this stage is shown in Figure 83. ------------
% ------------------------------------------------------------------------------------------

% ------------------------------  continuation  ------------------------------------------
   figure;                                % open the window for the new figure
   t=1:512;                               % define the time-points for signals
   subplot(2,1,2);                        % the 2nd sub-window will be used for signals
   plot(t,x,'b', t,real(ifft(X1)),'r');  % plot the signal x(n) in blue, x1(n) in red
   axis([0,512,0,40]);                    % to correct a little the window for the graph
   title('the original and filtered signals');  % add the title to the graph
% --------------  The result of the code on this stage is shown in Figure 84. ------------
```

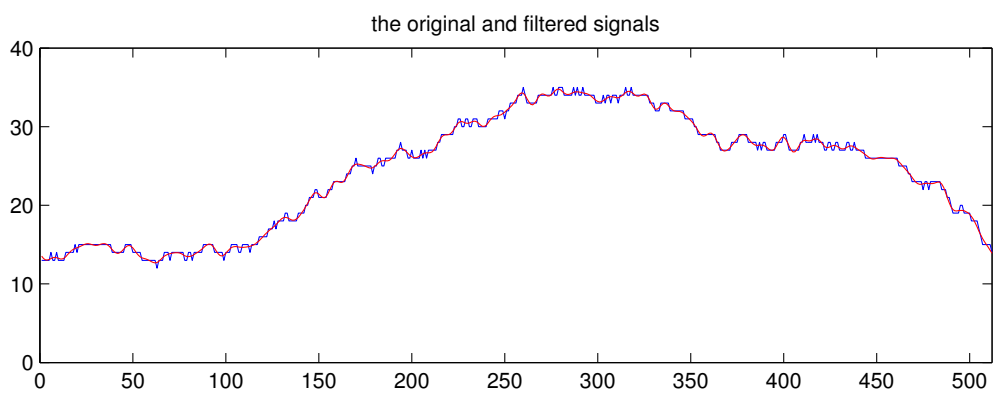


Fig. 84. The original signal together with the filtered signal.

```
% ------------------------------  continuation  ------------------------------------------
%5. The inverse DFT (h(n)) of the ideal filter; h(n) is called the impulse function of LPF
   h=zeros(1,N);                         % memory for the new variable h(k)
   h=real(ifft(H));                      % calculate the inverse 512-DFT of the LPF H(n)

   figure;                               % open the window for the new figure
   subplot(2,1,1);                       % the 1st sub-window will be used for h(k)
   plot(h);                              % plot the impulse function of the LPF H(n)
   axis([-10,520, -0.1, 0.3]);           % to correct a little the window for the graph
   title('inverse DFT h(n) of the Ideal LPF');       % add the title to the graph
```

```
   xlabel(' (a) ');                    % label the x-axis as "(a)"

   t1=-N/2:N/2-1;                      % calculate new time-point for the shifted h(k)
   subplot(2,1,2);                     % the 2nd sub-window will be used for shifted h(k)
   plot(t1,fftshift(h));               % plot the impulse function h(k) after shifting
   axis([-270,270, -0.1, 0.3]);        % to correct a little the window for the graph
   title('shifted inverse DFT h(n) of the Ideal LPF');    % add the title to the graph
   xlabel(' (b) ');                    % label the x-axis as "(a)"
% --------------  The result of the code on this stage is shown in Figure 85. ------------
% ----------------------------------------------------------------------------------------
```

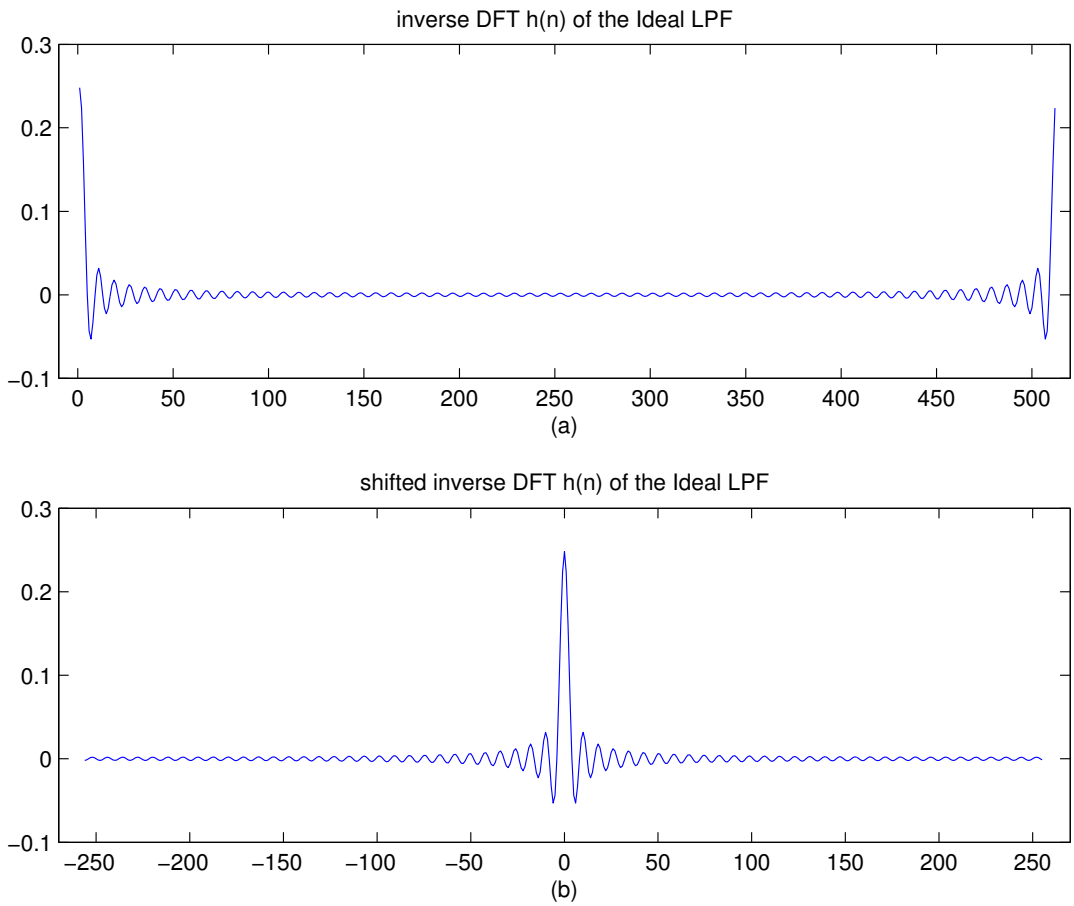


Fig. 85. The impulse function of the LPF (a) before and (b) after shifting.

```
% ==================================== end of the code ====================================
```

XVII. Fourier transform

In this section, we discuss the general concept of the Fourier transform, when not only periodic function can be uniquely represented in the frequency-domain.

It was mentioned in the lectures, that the coefficients of the Fourier series of the common periodic function $f(t)$ are not functions of the period T or fundamental frequency ω_0, i.e.,

$$C_0 \neq C_0(T), \quad C_n \neq C_n(T), \quad n = \pm 1, \pm 2, \ldots,$$

and

$$C_0 \neq C_0(\omega_0), \quad C_n \neq C_n(\omega_0), \quad n = \pm 1, \pm 2, \ldots.$$

The scaling of the function $f(t) \to f(\alpha t)$, when $\alpha \neq 0$, does not change the coefficients, i.e.,

$$C_n(f(t)) = C_n(f(\alpha t)), \quad n = \pm 1, \pm 2, \ldots.$$

For generalized functions like the delta function, this fact does not hold.

Example 1: Consider the periodic function $x(t)$ with the period $T = 2\pi$; thus $\omega_0 = 2\pi/T = 1$ rad/s. We assume that $x(t) = \delta(t)$ in the interval $[-T/2, T/2) = [-\pi, \pi)$. Then, the Fourier coefficients of $x(t)$ can be calculated as follows:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \delta(t) e^{-jn\omega_0 t}\, dt = \frac{1}{2\pi} = \frac{1}{T}, \tag{125}$$

for all $n = 0, \pm 1, \pm 2, \ldots$.

So, the delta function can be represented by the exponential basis functions as

$$\begin{aligned} \delta(t) &= \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{jnt} \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{+\infty} \cos(nt) = \frac{1}{T} + \frac{2}{T} \sum_{n=1}^{+\infty} \cos(n\omega_0 t) = \frac{1}{T} + \frac{2}{T} \sum_{n=1}^{+\infty} \cos\left(\frac{2\pi nt}{T}\right). \end{aligned}$$

Thus, the coefficients, d_n and phases, ϑ_n, of the cosine Fourier series for the periodic delta function are defined as

$$d_0 = 1/(2\pi), \quad d_n = 1/\pi, \quad \text{and} \quad \vartheta_n = 0, \quad n = 1, 2, \ldots.$$

Note, that

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-jnt} e^{jmt} dt = \delta_{n,m}$$

where $\delta_{n,m}$ is the Dirac symbol, $\delta_{n,m} = 1$ if $n = m$, and $\delta_{n,m} = 0$ if $n \neq m$.

A. Definition of the Fourier transform (FT)

For not periodic function, one can imagine that the function is considered on the large interval $[-T/2, T/2)$ and periodically extended on the real line. The fundamental frequency $\omega_0 = 2\pi/T$ becomes small when T grows. In the limit when $T \to \infty$, this period will cover the entire real line R and the concept of the Fourier series will be reduced to the new concept which we call *the Fourier transform.*

For an absolute (and square) integrable function $x(t)$ defined on the numeral line R,

$$\int_{-\infty}^{+\infty} |x(t)|^p dt < \infty, \quad p = 1, 2$$

we consider the Fourier transform,

$$(\mathcal{F} \circ x)(\omega) = X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, dt, \quad \omega \in (-\infty, +\infty). \tag{126}$$

The *inverse* Fourier transform is described as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t}\, d\omega, \quad t \in (-\infty, +\infty). \tag{127}$$

B. Properties of the FT

1. Duality takes place:

$$\begin{array}{ccc} x(t) & \to & X(\omega) \\ \downarrow & \swarrow & \downarrow \\ X(t) & \to & 2\pi x(-\omega) \end{array}$$

2. If for a function $x(t)$,

$$\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = 0,$$

then $x(t) \equiv 0$ almost everywhere, namely it follows that the zero response of the integrator

$$x(t) \to \int_{-\infty}^{t} x(\tau) d\tau = 0, \quad \forall t \in (-\infty, +\infty),$$

yields the input $x(\tau) = 0$ a.e. In other words, if $(x * u)(t) = 0$ then $x = 0$ a.e.

3.

$$|X(\omega)| \le \int_{-\infty}^{+\infty} |x(t)| dt, \quad \forall \omega \in (-\infty, +\infty).$$

4. For k-order derivative, $f^{(k)}(t)$, of the function f,

$$F[f^{(k)}](\omega) = (\mathcal{F} \circ f^{(k)})(\omega) = (j\omega)^k F(\omega), \quad k = 1, 2, 3, \dots .$$

Indeed, using the rule of integration by parts, we obtain the following:

$$\begin{aligned}
F[f'](\omega) &= \int_{-\infty}^{+\infty} f'(t) e^{-j\omega t} dt \\
&= f(t) e^{-j\omega t}|_{-\infty}^{+\infty} + j\omega \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt \\
&\qquad (f(t) \to 0,\ t \to \pm\infty, \text{ for absolute integrable function}) \\
&= j\omega \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt = j\omega F(\omega). \\
F[f^{(k)}](\omega) &= j\omega F[f^{(k-1)}](\omega) = (j\omega)^2 F[f^{(k-2)}](\omega) = \dots = (j\omega)^k F[f](\omega).
\end{aligned}$$

Application 1: We can use this property when solving the linear differential equations, for instance,

$$y^{(k)}(t) + b_{k-1} y^{(k-1)}(t) + \dots + b_1 y^{(1)}(t) + b y(t) = a x(t) + a_1 x^{(1)}(t).$$

Indeed, taking the Fourier transform of both sides, we obtain

$$\begin{gathered}
(j\omega)^k F[y] + b_{k-1}(j\omega)^{k-1} F[y] + \dots + b_1 (j\omega) F[y] + b F[y] = a F[x] + a_1 (j\omega) F[x], \\
F[y] \left((j\omega)^k + b_{k-1}(j\omega)^{k-1} + \dots + b_1 (j\omega) + b \right) = F[x] \left(a + a_1 (j\omega) \right).
\end{gathered}$$

Thus,

$$\begin{aligned}
Y(\omega) &= F[y](\omega) = H(\omega) F[x](\omega) = H(\omega) X(\omega), \\
H(\omega) &= \frac{a + a_1 (j\omega)}{b + b_1 (j\omega) + \dots + b_{k-1}(j\omega)^{k-1} + (j\omega)^k}
\end{aligned}$$

And the impulse response function of the linear time-invariant system described by the above differential equation is

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} H(\omega) e^{j\omega t} d\omega.$$

5. For a periodic and square integrable function $x(t)$ on the interval $[0, 2\pi]$ the following holds:

$$\sum_{-\infty}^{+\infty} |c_n|^2 = \frac{1}{2\pi} \int_0^{2\pi} |x(t)|^2 dt$$

where, the coefficients

$$c_n = \frac{1}{2\pi}\int\limits_0^{2\pi} x(t)e^{-jnt}\,dt, \quad n = 0, \pm 1, \ldots.$$

In the general case, when the function $x(t)$ is defined on the real line R

$$\int\limits_{-\infty}^{+\infty} |X(\omega)|^2\,d\omega = 2\pi \int\limits_{-\infty}^{+\infty} |x(t)|^2\,dt.$$

The following general formula holds. For given two functions $x_1(t)$ and $x_2(t)$

$$\int\limits_{-\infty}^{+\infty} X_1(\omega)\bar{X}_2(\omega)\,d\omega = 2\pi \int\limits_{-\infty}^{+\infty} x_1(t)\bar{x}_2(t)\,dt.$$

As a particular case, the distance between two functions $f(t)$ and $g(t)$ can be measured in the time domain as well as in the frequency domain (Parseval's theorem):

$$\int\limits_{-\infty}^{+\infty} |f(t) - g(t)|^2\,dt = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} |F(\omega) - G(\omega)|^2\,d\omega.$$

if use the previous result for $x_1(t) = x_2(t) = f(t) - g(t)$.

6. Consider the time-scaled version $x(at)$ of the signal

$$\begin{array}{ccc} x(t) & \to & x(at) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ X(\omega) & \to & \frac{1}{|a|}X(\frac{\omega}{a}) \end{array}$$

Indeed, if $a > 0$

$$\begin{aligned} \int\limits_{-\infty}^{+\infty} x(at)e^{-j\omega t}dt &= \int\limits_{-\infty}^{+\infty} x(at)e^{-j\frac{\omega}{a}at}dat \cdot \frac{1}{a} \\ &= \frac{1}{a}\int\limits_{-\infty}^{+\infty} x(t)e^{-j\frac{\omega}{a}t}dt \\ &= \frac{1}{a}X(\frac{\omega}{a}) \end{aligned}$$

In particular

$$\begin{array}{ccc} x(t) & \to & x(-t) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ X(\omega) & \to & X(-\omega) \end{array}$$

$$\begin{array}{ccccc} x(-\frac{1}{2}t) & \leftarrow & x(t) & \rightarrow & x(2t) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ 2X(-2\omega) & \leftarrow & X(\omega) & \rightarrow & \frac{1}{2}X(\frac{\omega}{2}) \end{array}$$

7. Consider the time-shifted version $x(t-t_0)$ of the signal

$$\begin{array}{ccl} x(t) & \rightarrow & x(t-t_0) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ X(\omega) & \rightarrow & e^{-j\omega t_0}X(\omega) = (|X(\omega)|, \vartheta(\omega) - \omega t_0) \end{array}$$

where $X(\omega) = (|X(\omega)|, \vartheta(\omega))$.

Indeed,

$$\begin{aligned} \int_{-\infty}^{+\infty} x(t-t_0)e^{-j\omega t}dt &= \int_{-\infty}^{+\infty} x(t-t_0)e^{-j\omega(t-t_0+t_0)}dt\cdot \\ &= \int_{-\infty}^{+\infty} x(t-t_0)e^{-j\omega(t-t_0)}e^{-j\omega t_0}dt \\ &= e^{-j\omega t_0}\int_{-\infty}^{+\infty} x(t-t_0)e^{-j\omega(t-t_0)}d(t-t_0) \\ &= e^{-j\omega t_0}X(\omega) \end{aligned}$$

In particular

$$\begin{array}{ccccc} x(t-1) & \leftarrow & x(t) & \rightarrow & x(t+1) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ e^{-j\omega}X(\omega) & \leftarrow & X(\omega) & \rightarrow & e^{j\omega}X(\omega) \end{array}$$

Due to the principle of duality, the following diagram holds

$$\begin{array}{ccccc} e^{jt\omega_0}x(t) & . & x(t-t_0) & \rightarrow & e^{-jt_0\omega}X(\omega) \\ \downarrow & . & & \swarrow \mathcal{F} & \\ X(\omega-\omega_0) & . & e^{-jt\omega_0}X(t) & \rightarrow & 2\pi x(\omega-\omega_0) \end{array}$$

8. Using property 7, we can find the Fourier transform of functions

$$y(t) = x(t)\sin(w_0 t), \quad y(t) = x(t)\cos(w_0 t).$$

Indeed

$$\begin{aligned} x(t)\sin(w_0 t) &= x(t)\frac{e^{jw_0 t} - e^{-jw_0 t}}{2j} \\ &= \frac{1}{2j}\left[x(t)e^{jw_0 t} - x(t)e^{-jw_0 t}\right] \end{aligned}$$

It directly results the diagram

$$\begin{array}{ccc} x(t) & \to & x(t)\sin(w_0 t) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ X(\omega) & \to & \dfrac{1}{2j}\left[X(\omega-\omega_0)-X(\omega+\omega_0)\right] \end{array}$$

Similarly, the following diagram is valid for function $x(t)\cos(w_0 t)$:

$$\begin{array}{ccc} x(t) & \to & x(t)\cos(w_0 t) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ X(\omega) & \to & \dfrac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right] \end{array}$$

9. For amplitude amplification, we have the following diagram for the Fourier transform:

$$\begin{array}{ccc} x(t) & \to & Ax(t)+B \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ X(\omega) & \to & AX(\omega)+B\cdot 2\pi\delta(\omega). \end{array}$$

10. The theorem of multiplication of spectra:

$$\begin{array}{ccccc} x(t), & h(t) & \Longrightarrow & (x*h)(t) \\ \downarrow & \downarrow & & \downarrow \\ X(\omega), & H(\omega) & \Longrightarrow & X(\omega)H(\omega) \end{array}$$

Indeed,

$$\int\limits_{-\infty}^{+\infty}(x*h)(t)e^{-j\omega t}dt = \int\limits_{-\infty}^{+\infty}\left(\int\limits_{-\infty}^{+\infty}x(t-\tau)h(\tau)\,d\tau\right)e^{-j\omega t}dt$$

In the integral, exponential function can be written as

$$e^{-j\omega t}=e^{-j\omega(t-\tau+\tau)}=e^{-j\omega(t-\tau)}e^{-j\omega\tau}$$

and therefore the last integral equals

$$\begin{aligned} &\int\limits_{-\infty}^{+\infty}\left(\int\limits_{-\infty}^{+\infty}x(t-\tau)h(\tau)\,d\tau\right)e^{-j\omega(t-\tau)}e^{-j\omega\tau}dt \\ &=\int\limits_{-\infty}^{+\infty}x(t-\tau)e^{-j\omega(t-\tau)}\,dt\left(\int\limits_{-\infty}^{+\infty}h(\tau)e^{-j\omega\tau}d\tau\right) \\ &=X(\omega)H(\omega). \end{aligned}$$

In particular, when $h(t)$ is the delta function,

$$\begin{array}{ccccccc} x(t), & \delta(t) & \Longrightarrow & (x*\delta)(t) & = & x(t) \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ X(\omega), & D(\omega) & \Longrightarrow & X(\omega)D(\omega) & = & X(\omega) \end{array}$$

Therefore, $D(\omega) = (\mathcal{F} \circ \delta)(\omega) \equiv 1$.

And, due to the principle of duality

$$\begin{array}{ccc} \delta(t) & \rightarrow & D(\omega) \equiv 1 \\ \downarrow & \swarrow & \downarrow \\ D(t) \equiv 1 & \rightarrow & 2\pi\delta(-\omega) = 2\pi\delta(\omega) \end{array}$$

Example 2: Consider the exponential function

$$x(t) = e^{-a|t|}, \quad a > 0.$$

The Fourier transform of this functions can be computed as follows:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} e^{-a|t|}[\cos\omega t - j\sin\omega t]dt \\ &= \int_{-\infty}^{+\infty} e^{-a|t|}\cos\omega t dt \\ &= 2\int_{0}^{+\infty} e^{-at}\cos\omega t dt = \frac{2a}{a^2+\omega^2} \end{aligned}$$

Due to the duality

$$x(t) = \frac{2a}{a^2+t^2} \longleftrightarrow 2\pi e^{-a|\omega|}.$$

In particular $a = 1$ case,

$$e^{-|t|} \longleftrightarrow \frac{2}{1+\omega^2}, \qquad \frac{2}{1+t^2} \longleftrightarrow 2\pi e^{-|\omega|}.$$

Example 3: Given $a > 0$, consider the exponential function

$$x(t) = e^{-at}u(t), \quad t \in (-\infty, +\infty).$$

Then

$$\begin{aligned} X(\omega) &= \int_{0}^{+\infty} e^{-at}e^{-j\omega t}dt \\ &= \int_{0}^{+\infty} e^{-(a+j\omega)t}dt = \frac{1}{a+j\omega} \end{aligned}$$

and due to the duality,

$$x(t) = \frac{1}{a+jt} \longleftrightarrow 2\pi e^{a\omega} u(-\omega).$$

Example 4: Consider the rectangle function on the interval $[-T_0, T_0]$,

$$x(t) = rect\left(\frac{t}{2T_0}\right) = u(t+T_0) - u(t-T_0).$$

Then,

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_{-T_0}^{T_0} e^{-j\omega t} dt \\ &= \frac{e^{j\omega T_0} - e^{-j\omega T_0}}{j\omega} \\ &= \frac{2j \sin \omega T_0}{j\omega} \\ &= 2T_0 \frac{\sin \omega T_0}{\omega T_0} = 2T_0 \operatorname{sinc}(\omega T_0) \end{aligned}$$

where we denote

$$\operatorname{sinc}(t) = \frac{\sin(t)}{t}, \ \ t \neq 0, \qquad \operatorname{sinc}(0) = 1.$$

As a particular case, we obtain

$$\operatorname{rect}(t) \longleftrightarrow \operatorname{sinc}(\omega/2),$$

Note that $\operatorname{sinc}(\omega)$ function is not absolute integrable, i.e.,

$$\int_{-\infty}^{+\infty} |\operatorname{sinc}(\omega)| \, d\omega = \infty.$$

Figure 86 shows the $\operatorname{sinc}(\omega)$ function.
Example 5: Consider the integrator

$$x(t) \to y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau.$$

The response function of the integrator is $u(t)$, and the following diagram valid

$$\begin{array}{ccccc} x(t), & u(t) & \Longrightarrow & (x * u)(t) \\ \downarrow & \downarrow & & \downarrow \\ X(\omega), & U(\omega) & \Longrightarrow & X(\omega)U(\omega) \end{array}$$

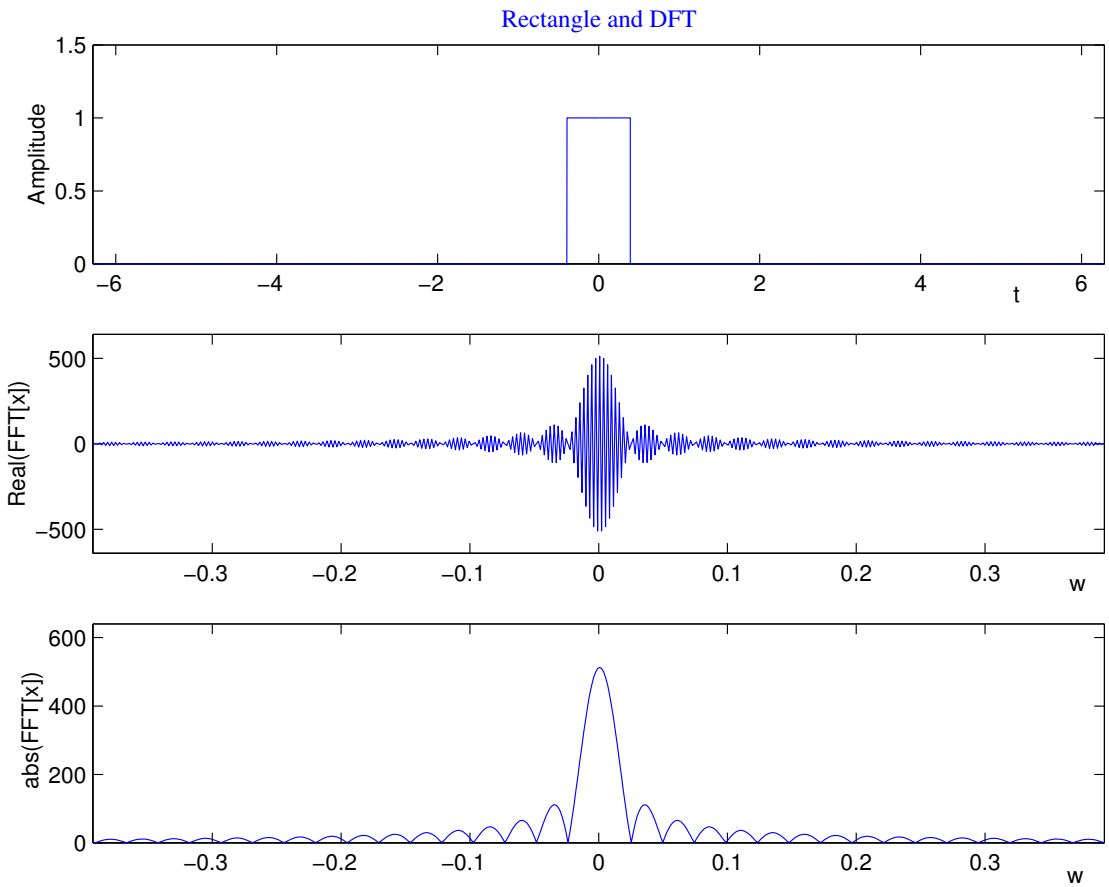


Fig. 86. The rectangle function, its Fourier transform, $\text{sinc}(\omega)$, and $|\text{sinc}(\omega)|$ funstion.

The Fourier transform of the unit step function is

$$(\mathcal{F} \circ u)(\omega) = U(\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Therefore,

$$\begin{aligned} X(\omega)U(\omega) &= X(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] \\ &= \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega). \end{aligned}$$

We now consider the Fourier transform over functions (non periodic) defined on the real line $R = (-\infty, +\infty)$.

Example 6: Consider first the periodic pulse function defined on the interval $[0, T]$ as

$$\begin{aligned} x(t) &= rect(t) \to z(t) = x\left(t - \frac{1}{2}\right) \\ z(t) &\to y(t) = z\left(\frac{t}{T}\right) = x\left(\frac{t}{T} - \frac{1}{2}\right) \end{aligned}$$

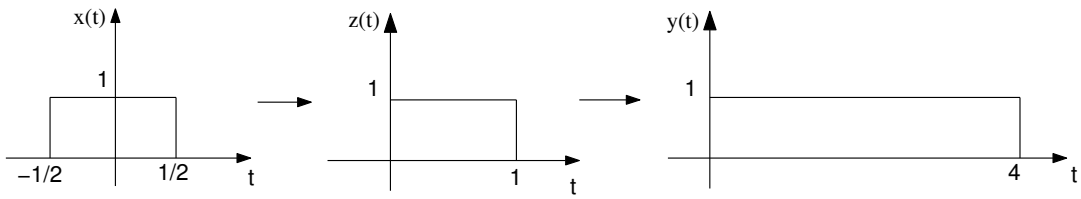


Fig. 87. Transformation of the unit rectangle.

The Fourier transforms of above functions are described by the diagram as

$$\boxed{\begin{array}{ccccccccc} x(t) & \to & z(t) & = & x(t-\frac{1}{2}) & \to & z(\frac{t}{T}) & = & x\left(\frac{t}{T}-\frac{1}{2}\right) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ X(\omega) & \to & Z(\omega) & = & X(\omega)e^{-j\frac{1}{2}\omega} & \to & TZ(T\omega) & = & TX(T\omega)e^{-j\frac{1}{2}\omega T} \end{array}}$$

We also know that the Fourier transform of $rect(t)$ is the sinc function

$$X(\omega) = \text{sinc}\left(\frac{\omega}{2}\right)$$

therefore, we obtain

$$\mathcal{F} : x\left(\frac{t}{T} - \frac{1}{2}\right) \to T\text{sinc}\left(\frac{T\omega}{2}\right) e^{-j\frac{1}{2}\omega T}.$$

Example 7: Consider function

$$x(t) = e^{-a|t|}, \quad a > 0.$$

To find the Fourier transform of $x(t)$, we perform the following calculations

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\, dt$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} e^{-a|t|}[\cos\omega t - j\sin\omega t]\,dt \\
&= \int_{-\infty}^{\infty} e^{-a|t|}\cos\omega t\,dt = 2\int_{0}^{\infty} e^{-at}\cos\omega t\,dt
\end{aligned}
$$

Therefore, by using two times the integration by parts we find the following:

$$
\begin{aligned}
\int_{0}^{\infty} e^{-at}\cos\omega t\,dt &= -\frac{1}{a}\int_{0}^{\infty}\cos(\omega t)de^{-at} \\
&= -\frac{1}{a}\left[\cos(\omega t)e^{-at}\,\Big|_0^{\infty} - \int_{0}^{\infty} e^{-at}d\cos\omega t\right] \\
&= -\frac{1}{a}\left[0 - 1 - \omega\int_{0}^{\infty} e^{-at}\sin(\omega t)dt\right] \\
&= -\frac{1}{a}\left[-1 - \omega\left[-\frac{1}{a}\int_{0}^{\infty}\sin(\omega t)de^{-at}\right]\right] \\
&= -\frac{1}{a}\left[-1 + \frac{\omega}{a}\left[\sin(\omega t)e^{-at}\,\Big|_0^{\infty} - \int_{0}^{\infty} e^{-at}d\sin(\omega t)\right]\right] \\
&= -\frac{1}{a}\left[-1 + \frac{\omega^2}{a}\left[\int_{0}^{\infty} e^{-at}\cos(\omega t)dt\right]\right]
\end{aligned}
$$

Denoting by

$$A = \int_{0}^{\infty} e^{-at}\cos\omega t\,dt$$

we obtain

$$A = -\frac{1}{a}\left[-1 + \frac{\omega^2}{a}A\right] \to aA = 1 - \frac{\omega^2}{a}A,$$

which results in

$$A = \frac{a}{a^2+\omega^2}.$$

Finally, we obtain

$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}\,dt = 2A = \frac{2a}{a^2+\omega^2}.$$

$x(t)$ is symmetric and its Fourier transform is real.

Example 8: For signal

$$x(t) = e^{-at}u(t), \quad a > 0,$$

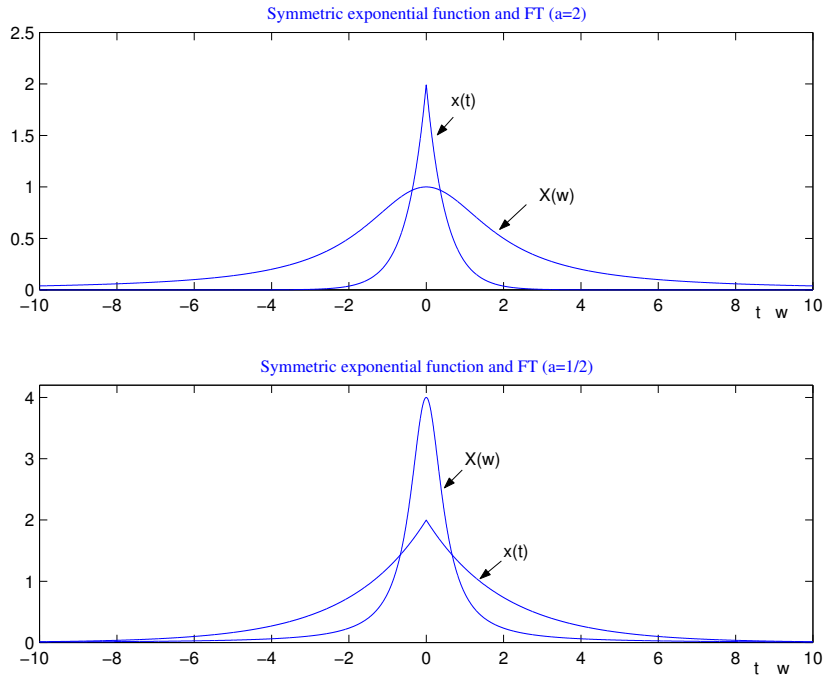


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Fig. 88. Laplacian functions $x(t) = 2e^{-a|t|}$ and their Fourier transforms for $a = 2$ and $a = 0.5$.

the Fourier transform can be calculated as follows

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} \left(e^{-at}u(t)\right) e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-at-j\omega t} dt \\ &= \int_0^{\infty} e^{-(a+j\omega)t} dt = -\frac{1}{a+j\omega} e^{-(a+j\omega)t}\Big|_o^{\infty} \\ &= \frac{1}{a+j\omega} \end{aligned}$$

So, the power spectrum of the signal

$$|X(\omega)|^2 = \frac{1}{a+j\omega} \cdot \frac{1}{a-j\omega} = \frac{1}{a^2+\omega^2}$$

if $Re\, a \neq 0$.

Example 9: Consider the Fourier transform of the Fourier series of the periodic function $x(t)$ with period T. Taking the Fourier series of the function

$$x(t) = \sum_{n \in Z} c_n e^{j\omega_0 nt}, \quad \omega_0 = 2\pi/T,$$

we can write the Fourier transform as

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \left[\sum_{n \in Z} c_n e^{j\omega_0 nt} \right] e^{-j\omega t} dt \\ &= \sum_{n \in Z} c_n \int_{-\infty}^{\infty} e^{j\omega_0 nt} e^{-j\omega t} dt \\ &= \sum_{n \in Z} c_n \int_{-\infty}^{\infty} e^{-jt(\omega - \omega_0 n)} dt \\ &= \sum_{n \in Z} c_n 2\pi \delta(\omega - \omega_0 n) \\ &= 2\pi \sum_{n \in Z} c_n \delta(\omega - n\omega_0) \end{aligned}$$

B. Fourier transform of periodic functions

We now consider main properties of the Fourier transform of a periodic function, namely, the representation of the Fourier transform. Let T be a fundamental period of a function $x(t)$,

$$x(t) = x(t+T), \quad t \in R.$$

Then, defining the function (one period)

$$\widehat{x}(t) = \begin{cases} x(t), & t \in [0, T) \\ 0, & t \notin [0, T) \end{cases}$$

we can write function $x(t)$ as the following set of shifted versions of $\widehat{x}(t)$, i.e.

$$x(t) = \{..., \widehat{x}(t+T), \widehat{x}(t), \widehat{x}(t-T), ...\}$$

or

$$\begin{aligned} x(t) &= \sum_{k \in Z} \widehat{x}(t - kT) = \sum_{k \in Z} \widehat{x}(t) * \delta(t - kT) \\ &= \widehat{x}(t) * \sum_{k \in Z} \delta(t - kT). \end{aligned}$$

We know, the Fourier series of the periodic infinite-pulse (delta) function can be written as

$$\sum_{k\in Z}\delta(t-kT)=\sum_{k\in Z}\frac{1}{T}e^{-jk\omega_0 t},\quad \omega_0=2\pi/T.$$

Therefore

$$x(t)=\widehat{x}(t)*\sum_{k\in Z}\frac{1}{T}e^{-jk\omega_0 t}.$$

Then, denoting by

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt,\\ \widehat{X}(\omega) &= \int_{-\infty}^{\infty} \widehat{x}(t)e^{-j\omega t}\,dt, \end{aligned}$$

and using the fact that the convolution is a operation of multiplication in the Fourier domain, we obtain the following:

$$\begin{array}{ccccc} x(t) & = & \widehat{x}(t) & * & \frac{1}{T}\sum_{k\in Z} e^{-jk\omega_0 t}\\ \downarrow & & \downarrow & \downarrow & \downarrow\\ X(\omega) & = & \widehat{X}(\omega) & \cdot & \frac{1}{T}2\pi\sum_{k\in Z}\delta(\omega-k\omega_0) \end{array}$$

So

$$\begin{aligned} X(\omega) &= \frac{2\pi}{T}\sum_{k\in Z}\widehat{X}(\omega)\delta(\omega-k\omega_0)\\ &= \omega_0\sum_{k\in Z}\widehat{X}(k\omega_0)\delta(\omega-k\omega_0) \end{aligned}$$

and the Fourier transform $X(\omega)$ of the periodic function has nonzero values only at integer frequencies $k\omega_0$.

Example 10: In the case, when $\widehat{x}(t)=\delta(t)$, we have

$$\begin{array}{ccc} \widehat{x}(t) & = & \delta(t)\\ \downarrow & & \downarrow\\ \widehat{X}(\omega) & = & 1 \end{array}$$

$$\begin{array}{ccc} x(t) & = & \sum_{n\in Z}\widehat{x}(t-nT)\\ \downarrow & & \downarrow\\ X(\omega) & = & \omega_0\sum_{n\in Z}\delta(\omega-n\omega_0) \end{array}$$

where the fundamental frequency $\omega_0=2\pi/T$.

C. Frequency characteristics of a LTI system

Let H be a linear time-invariant system with the impulse response function $h(t)$. We consider the input of the system, that have the form

$$x(t) = e^{j\omega t}, \quad \text{given } \omega \in R.$$

Then, by definition of the linear convolution,

$$\begin{aligned} x(t) \quad \to \quad (x * h)(t) &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) e^{j\omega(t-\tau)}\, d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) e^{j\omega t} e^{-j\omega\tau}\, d\tau \\ &= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau}\, d\tau \\ &= e^{j\omega t} H(\omega) \qquad \left(\frac{Y(\omega)}{X(\omega)} = H(\omega)\right) \end{aligned}$$

So

$$\begin{aligned} e^{j\omega t} \quad &\to \quad e^{j\omega t} H(\omega) \\ a_1 e^{j\omega_1 t} + a_2 e^{j\omega_2 t} \quad &\to \quad a_1 e^{j\omega_1 t} H(\omega_1) + a_2 e^{j\omega_2 t} H(\omega_2) \end{aligned}$$

and using the Fourier transform representation, we obtain

$$\begin{aligned} x(t) \quad &= \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega \\ &\to \int_{-\infty}^{\infty} X(\omega) H[e^{j\omega t}]\, d\omega \\ y(t) \quad &= \int_{-\infty}^{\infty} X(\omega) H(\omega) e^{j\omega t}\, d\omega \end{aligned}$$

which is the theorem of multiplication of spectra,

$$y(t) = x(t) * h(t) \to X(\omega) H(\omega) = Y(\omega).$$

XVIII. The Laplace transform *EE-3424*

The most common system in electrical engineering is circuit analysis that can be done in the time domain or/and in the frequency domain. Because of powerful methods of the linear algebra, the circuit analysis is usually done in the frequency domain. The analysis of differential equations that describe the relations between output and input as well as between different components of the circuit is reduced to algebraic equations. Many simple circuit elements such as resistors, conductors, capacitors, diodes, voltage and current sources can be modeled as linear devices (systems) and described by transfer functions. The frequency analysis uses mainly the Laplace transform. The technique of the Fourier transform cannot be applied in many cases for signals or system characteristics for which the integral does not converge.

In many practical applications, we need solve differential equations with functions $f(t)$ that grow fast when t tends to $\pm\infty$. The functions are not absolute integrable, and therefore, the Fourier transform cannot be applied directly to simplify the differential equations in the frequency domain. But if functions do not grow faster than exponential functions, then a new Fourier integral can be evaluated. We face with a situation when a function $f(t)$ is not integrable but becomes such after multiplying by exponential function $e^{-\alpha t}$, for a certain real (or complex) number a. For example, the function $f(t) = e^{2t}u(t)$ is not absolute integrable, but the function $f(t)e^{-3t} = e^{2t}e^{-3t}u(t) = e^{-t}u(t)$ is integrable. In other words, for such case the following integral exists

$$\mathcal{F}\left[f(t)e^{-\alpha t}\right] = \int\limits_{-\infty}^{\infty} \left[f(t)e^{-\alpha t}\right] e^{-j\omega t}dt = \int\limits_{-\infty}^{\infty} f(t)e^{-(\alpha+j\omega)t}dt \tag{128}$$

we can even derive the domain of α for which this integral is defined. Indeed, let $\alpha = \alpha_1 + j\alpha_2$, then the integral in the above equation can be written as

$$\int\limits_{-\infty}^{\infty} f(t)e^{-(\alpha_1+j\alpha_2+j\omega)t}dt = \int\limits_{-\infty}^{\infty} f(t)e^{-\alpha_1 t}e^{-j(\alpha_2+\omega)t}dt$$

and if the function $f(t)e^{-\alpha_1 t}$ is absolute integrable then the transform $\mathcal{F}\left[f(t)e^{-\alpha t}\right]$ exists.

Example 1: Given number p (on the imaginary vertical $p = j\omega$), we consider the following transform for the unit function $u(t)$:

$$\begin{aligned} f(t) &= u(t) \to \int_{-\infty}^{\infty} u(t)e^{-pt}dt = \int_0^{\infty} e^{-pt}dt \\ &= \int_0^{\infty} \frac{1}{-p}de^{-pt} =^{?} -\frac{1}{p}e^{-pt}\,|_0^{\infty} \to (p = j\omega) \to^{?} \frac{1}{j\omega} - \frac{1}{j\omega}e^{-j\omega t}|_{t=\infty} \end{aligned}$$

with no result, because the last integral cannot be evaluated. Let us now multiply the function $f(t)$ by a factor $e^{-\alpha t}$, where $\alpha = \alpha_1 + j\alpha_2$ is a complex number to be defined. The Fourier transform of the new function $g(t) = f(t)e^{-\alpha t}$ is defined as

$$\begin{aligned} g(t) &\to G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \underline{u(t)e^{-\alpha t}}e^{-j\omega t}dt \\ &= \int_0^{\infty} e^{-(\alpha_1+j\alpha_2)t}e^{-j\omega t}dt = \int_0^{\infty} e^{-j(\alpha_2+\omega)t}e^{-\alpha_1 t}dt \quad \exists \end{aligned}$$

because

$$\left|e^{-j(\alpha_2+\omega)t}e^{-\alpha_1 t}\right| = e^{-\alpha_1 t} \to 0, \quad t \to \infty,$$

when $\alpha_1 > 0$. We can therefore define integral $G(\omega)$ for any $\alpha_1 > 0$, or Re $\alpha > 0$.

To understand the concept of the Laplace transform, we first describe the response of a linear time invariant system L to a complex exponential signal

$$x(t) = x_s(t) = Xe^{st}, \quad t \in R, \quad (s \in R \text{ or } C^2)$$

where s is a real or complex number. The response is defined as follows

$$\begin{aligned} x_s(t) \quad &\to \quad y(t) = x_s(t) * h(t) = \int\limits_{-\infty}^{\infty} h(\tau)x_s(t-\tau)d\tau \\ &= \quad \int\limits_{-\infty}^{\infty} h(\tau)Xe^{s(t-\tau)}d\tau = Xe^{st}\underline{\int\limits_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau} = Xe^{st}H(s) = x_s(t)H(s) \end{aligned}$$

where the coefficient $H(s)$ is defined by

$$H(s) = \int\limits_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau. \tag{129}$$

Similarly, the response of the system to a linear combination of complex exponentials $x_{s_1}(t)$ and $x_{s_2}(t)$ is expressed as a linear combination of exponentials with the coefficients multiplied by $H(s_1)$ and $H(s_2)$. In other words,

$$\underline{X_1}x_{s_1}(t) + \underline{X_2}x_{s_2}(t) \to \underline{H(s_1)X_1}x_{s_1}(t) + \underline{H(s_2)X_2}x_{s_2}(t).$$

We recall here that the Fourier transform of a time-continuous function is a linear combination (finite or infinite) of a complex sinusoidal functions of the form $e^{j\omega t}$, where ω is a frequency (real variable)

$$F(\omega) = \int\limits_{-\infty}^{\infty} f(t)e^{-j\omega t}dt.$$

A. Definition of the Laplace transform

Consider the function $f(t)$ of the real variable $t \in (-\infty, +\infty)$. The transform

$$F(p) = \int\limits_{0}^{\infty} f(t)e^{-pt}dt, \;\; p = p_1 + jp_2, \;\; p_1, p_2 \in (-\infty, +\infty), \tag{130}$$

is called the *Laplace transform* (or unilateral Laplace transform). The set of p for which the Laplace transform can be defined is called a *region of convergence* for which we will use notation ROC(F) or (ROC).

The transformation

$$\mathcal{L} : f(t) \to F(p)$$

of the function of real variable t to the complex function of complex variable p, is defined for functions $f(t)$ of *exponential order* on $[0,\infty)$, i.e., for which there exist positive number M and a such that

$$|f(t)| \le Me^{at}u(t), \quad t \in (-\infty, +\infty).$$

The integral in (130) can be defined if $Re\, p = p_1 > a$.

Indeed, the following calculations hold:

$$\begin{aligned} \left|\int_0^\infty f(t)e^{-pt}dt\right| &\le M\int_0^\infty e^{at}e^{-p_1 t}dt = M\int_0^\infty e^{-(p_1-a)t}dt \\ &= M\frac{1}{-(p_1-a)}\int_0^\infty de^{-(p_1-a)t} = M\frac{1}{p_1-a} \end{aligned}$$

when $p_1 > a$.

The Laplace transform exists for any piecewise continuous function of exponential order.

Example 2: For the unit function $u(t)$, the Laplace transform is

$$U(p) = \int_0^\infty u(t)e^{-pt}dt = \int_0^\infty e^{-pt}dt = \frac{1}{-p}\int_0^\infty de^{-pt} = \frac{1}{p}$$

when $Re\, p > 0$.

Example 3: For the exponential function

$$f(t) = e^{\alpha t}u(t),$$

the Laplace transform is calculated as follows:

$$F(p) = \int_0^\infty f(t)e^{-pt}dt = \int_0^\infty e^{\alpha t}e^{-pt}dt = \frac{1}{p-\alpha}$$

when $Re\, p > Re\, \alpha$.

As a result, we obtain the following

$$\mathcal{L} : u(t) \to \frac{1}{p}, \qquad Re\, p > 0,$$

and for the cosine function the Laplace transform is defined as follows

$$\begin{array}{lcl} \cos(\omega t)u(t) & = & \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}]u(t) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ \mathcal{L}[\cos(\omega t)u(t)] & = & \frac{1}{2}\left[\frac{1}{p-j\omega} + \frac{1}{p+j\omega}\right] = \left[\frac{p}{p^2+\omega^2}\right] \end{array}$$

if $Re\, p > |Re\,(j\omega)| = 0$.

We obtain a similar result for the sine function

$$\begin{array}{lcl} \sin(\omega t)u(t) & = & \frac{1}{2j}[e^{j\omega t} - e^{-j\omega t}]u(t) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ \mathcal{L}[\sin(\omega t)u(t)] & = & \frac{1}{2j}\left[\frac{1}{p-j\omega} - \frac{1}{p+j\omega}\right] = \left[\frac{\omega}{p^2+\omega^2}\right]. \end{array}$$

B. Properties of the Laplacian transform

1. *(Linearity)*
If

$$x(t) \to X(p), \quad y(t) \to Y(p),$$

then

$$x(t) + ky(t) \to X(p) + kY(p),$$

for any constant $k \in R$, but p from the intersection of the ROCs.

2. *(Delay)*
Let us define the shift function

$$x_{t_0}(t) = x(t-t_0)u(t-t_0) = \begin{cases} 0, & t < t_0, \\ x(t-t_0), & t \geq t_0. \end{cases}$$

Then, the following calculations hold:

$$\begin{array}{ccc} x(t) & \to & x_{t_0}(t) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ X(p) & \to & X(p)e^{-pt_0} \end{array}$$

for any real t_0.
Indeed,

$$X_{t_0}(p) = \int_0^\infty x_{t_0}(t)e^{-pt}dt = \int_{t_0}^\infty x(t-t_0)e^{-pt}dt = \int_0^\infty x(t)e^{-p(t+t_0)}dt = X(p)e^{-pt_0}$$

for p such that $Re\, p > \alpha$.

Example 4: The piecewise function

$$f(t) = \begin{cases} 0, & t < a, \\ n, & na \leq t < (n+1)a, \ n = 1, 2, 3, ... \end{cases}$$

can be written as

$$f(t) = [u(t-a) + u(t-2a) + u(t-3a) + ...]$$

The following diagram holds for the Laplace transforms

$$\begin{array}{ccccccccc} f(t) & = & u(t-a) & + & u(t-2a) & + & u(t-3a) & + & ... \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} & & \downarrow \mathcal{L} & & \downarrow \mathcal{L} & & \\ F(p) & = & U(p)e^{-pa} & + & U(p)e^{-2pa} & + & U(p)e^{-3pa} & + & ... \end{array}$$

and since $U(p) = 1/p$, we obtain

$$\begin{aligned} F(p) &= \frac{1}{p}[e^{-pa} + e^{-2pa} + e^{-3pa} + ...] \\ &= \frac{1}{p}[(e^{-pa}) + (e^{-pa})^2 + (e^{-pa})^3 + ...] = \frac{1}{p}\frac{e^{-pa}}{1 - e^{-pa}}, \quad \text{if } p \in ROC(F). \end{aligned}$$

Region of convergence of the above geometrical progression is defined as the set of numbers p for which $|e^{-pa}| < 1$. Since $p = p_1 + jp_2$, then $|e^{-pa}| = |e^{-p_1 a}| \cdot |e^{-jp_2 a}| = |e^{-p_1 a}| < 1$. Since $a > 0$, the $p_1 > 0$ and ROC(F) is the set $\{p; Re\, p > 0\}$.

3. *(Derivative)*

$$\begin{array}{ccc} x(t) & \to & x'(t) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ X(p) & \to & pX(p) - x(0) \end{array} \qquad (Re\, p > a)$$

Indeed,

$$\mathcal{L}[x'](p) = \int_0^\infty x'(t)e^{-pt}dt = x(t)e^{-pt}|_0^\infty + p\int_0^\infty e^{-pt}x(t)dt = -x(0) + pX(p).$$

Similarly, in the general $n \geq 1$ case, we obtain

$$\mathcal{L} : x^{(n)}(t) \to p^n \left[X(p) - \frac{x(0)}{p} - \frac{x'(0)}{p^2} - \frac{x^{(2)}(0)}{p^3} - ... - \frac{x^{(n-1)}(0)}{p^n}\right].$$

Application

Consider the solution of the differential equation with constant coefficients

$$\begin{gathered} y^{(n)}(t) + a_1 y^{(n-1)}(t) + a_2 y^{(n-2)}(t) + ... + a_n y(t) = bx(t), \\ y(0) = y'(0) = y^{(2)}(0) = ... = y^{(n-1)}(0) = 0. \end{gathered}$$

Using the Laplace transform, we obtain

$$Y(p)[p^n + a_1 p^{n-1} + a_2 p^{n-2} + \cdots + a_{n-1}p' + a_n] = bX(p),$$

or

$$Y(p) = \frac{bX(p)}{P_n(p)} = \frac{bX(p)}{p^n + a_1 p^{n-1} + a_2 p^{n-2} + ... + a_n}.$$

4. *(Integrator)*

$$\begin{array}{ccc} x(t) & \to & \int_0^t x(\tau)d\tau \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ X(p) & \to & \dfrac{X(p)}{p} \end{array} \qquad (Re\, p > a)$$

Indeed, if $x(t) \to y(t)$, then

$$\begin{array}{lcll} y'(t) & = & x(t), & y(0)=0 \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} & \\ pY(p)-y(0) & = & X(p) & \end{array}$$

4a. *(n Integrators)*

$$\begin{array}{lll} x(t) & \to & \int\limits_0^t dt_1 \int\limits_0^{t_1} dt_2 \ldots \int\limits_0^{t_{n-1}} x(t_n)dt_n \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ X(p) & \to & \dfrac{X(p)}{p^n} \end{array} \qquad (Re\, p > a)$$

5. *(Convolution)*

$$\begin{array}{lll} x(t), & y(t), & x(t) * y(t) \\ \downarrow \mathcal{L} & \downarrow \mathcal{L} & \downarrow \mathcal{L} \\ X(p), & Y(p), & X(p)Y(p) \end{array} \qquad (Re\, p > \max\{a,b\})$$

if

$$|x(t)| \le Me^{at}u(t), \quad |y(t)| \le Ne^{bt}u(t), \qquad t>0.$$

Example 5: Let us find the original function of the following Laplace transform

$$F(p) = \frac{wp}{(p^2+\omega^2)^2}.$$

Then, we can compose the following diagram:

$$\begin{array}{lll} \cos(\omega t)u(t), & \sin(\omega t)u(t), & [\cos(\omega t)u(t)] * [\sin(\omega t)u(t)] = \dfrac{t}{2}\sin(\omega t) \\ \downarrow \mathcal{L} & \downarrow \mathcal{L} & \downarrow \mathcal{L} \\ \left[\dfrac{p}{p^2+\omega^2}\right], & \left[\dfrac{\omega}{p^2+\omega^2}\right], & \dfrac{wp}{(p^2+\omega^2)^2} = F(p) \end{array}$$

if $Re\, p > |Im\, \omega|$, or $Re\, p > 0$ if ω is real.

Therefore, the convolution of these cosine and sine waves can be calculated as follows:

$$\begin{aligned} f(t) &= \int\limits_0^t \sin(\omega[t-\tau])\cos(\omega\tau)d\tau = \frac{1}{2}\int\limits_0^t [\sin(\omega t) + \sin(\omega[t-2\tau])]d\tau \\ &= \frac{1}{2}\int\limits_0^t \sin(\omega t)d\tau + \frac{1}{2}\int\limits_{-t}^t \sin(\omega\tau)d\tau = \frac{t}{2}\sin(\omega t). \end{aligned}$$

6. *(Inverse formula)*

$$f(t) = \frac{1}{2\pi j} \int\limits_{x-j\infty}^{x+j\infty} F(p) e^{pt} dp, \qquad x > a.$$

where x is real. The vertical line $Re\, p = x$ is from the region of the existence of the Laplace transform of $f(t)$.

7. *(Delta function)*
For any real number $t_0 > 0$,

$$\begin{array}{ccc} \delta(t) & \to & \delta(t-t_0) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ 1 & \to & e^{-pt_0}. \end{array}$$

Indeed,

$$\int\limits_0^\infty \delta(t-t_0) e^{-pt} dt = e^{-pt}|_{t=t_0} = e^{-pt_0}.$$

Example 6: For the ramp function

$$x(t) = r(t)u(t),$$

the following diagram takes place

$$\begin{array}{ccccccc} e^{-at}u(t) & = & 1u(t) & = & x'(t) & \leftarrow & x(t) \\ \downarrow \mathcal{L} & (a=0) & \downarrow \mathcal{L} & & \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ \dfrac{1}{p+a} & = & \dfrac{1}{p} & = & pX(p) & \leftarrow & X(p) \end{array}$$

and, therefore,

$$X(p) = \frac{1}{p^2}, \quad Re\, p > 0.$$

8. *(Time scaling)*
For any real number $k > 0$,

$$\begin{array}{ccc} f(t) & \to & f(kt) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ F(p) & \to & \dfrac{1}{k} F(\dfrac{p}{k}) \end{array}$$

Indeed,

$$\int\limits_0^\infty f(kt) e^{-tp} dt$$

$$= \int\limits_0^\infty f(kt) e^{-p\frac{1}{k}kt} dkt \frac{1}{k}$$

$$\begin{aligned} &= \frac{1}{k}\int_0^\infty f(t)e^{-\frac{p}{k}t}dt \\ &= \frac{1}{k}F(\frac{p}{k}). \end{aligned}$$

Remark that $k > 0$ and we do not define the Laplace transform over the function $f(-t)$, when t is positive.

Example 7: Let us consider the time-transformation version of the function $\sin(t)u(t)$

$$\sin(t) \to \sin(\omega t - t_0)u(\omega t - t_0) = \sin\left(\omega\left[t - \frac{t_0}{\omega}\right]\right) u\left(\omega\left[t - \frac{t_0}{\omega}\right]\right).$$

Then, the Laplace transform can be calculated as

$$\begin{array}{ccc} \sin(\omega t)u(t) & \to & \sin\left(\omega(t - \frac{t_0}{\omega})\right) u\left(\omega(t - \frac{t_0}{\omega})\right) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ \dfrac{\omega}{p^2+\omega^2} & \to & \dfrac{\omega}{p^2+\omega^2}e^{-p\frac{t_0}{\omega}} \end{array}$$

or, we can use the calculation by the following diagram:

$$\begin{array}{ccc} \sin(t)u(t) & \to \sin(t-t_0)\,u(t-t_0) & \to \sin(t\omega - t_0)\,u(t\omega - t_0) \\ \downarrow \mathcal{L} & \downarrow \mathcal{L} & \downarrow \mathcal{L} \\ \dfrac{1}{p^2+1} & \to \dfrac{1}{p^2+1}e^{-pt_0} & \to \dfrac{1}{\omega}\dfrac{1}{(\frac{p}{\omega})^2+1}e^{-\frac{p}{\omega}t_0} \end{array}$$

9. *(Derivative of the Laplace image)*
The following diagram takes place

$$\begin{array}{ccc} F(p) & \to & -F'(p) \\ \uparrow \mathcal{L} & & \uparrow \mathcal{L} \\ f(t) & \to & tf(t) \end{array}$$

Indeed,

$$\begin{aligned} F'(p) &= \frac{d}{dp}\int_0^\infty f(t)e^{-pt}dt \\ &= -\int_0^\infty f(t)te^{-pt}dt \\ &= -\int_0^\infty [tf(t)]e^{-pt}dt \\ &= -\mathcal{L}[tf(t)](p). \end{aligned}$$

Example 8: Let us consider the function $f(t) = \sin(t)$, then

$$\begin{array}{ccccc} \sin(\omega t) & \to & \cos(\omega t) * \sin(\omega t) & = & \frac{t}{2}\sin(\omega t) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ \dfrac{\omega}{p^2+\omega^2} & & \dfrac{wp}{(p^2+\omega^2)^2} & = & -\frac{1}{2}\left(\dfrac{\omega}{p^2+\omega^2}\right)' \end{array}$$

if $Re\, p > 0$.

Similarly, we obtain

$$\begin{array}{ccccc} \cos(\omega t)u(t) & \to & t \cdot \cos(\omega t)u(t) & & \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} & & \\ \dfrac{p}{p^2+\omega^2} & \to & -\left(\dfrac{p}{p^2+\omega^2}\right)' & = & \dfrac{\omega^2-p^2}{(\omega^2+p^2)^2} \end{array}$$

since

$$-\left(\frac{p}{p^2+\omega^2}\right)' = \frac{1\cdot(p^2+\omega^2)-2p^2}{(p^2+\omega^2)^2}.$$

Another example of using this property is described by the following diagram:

$$\begin{array}{ccccccccc} tu(t) & \to & t^2u(t) & & t^3u(t) & & \to & & t^4u(t) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} & & \downarrow \mathcal{L} & & & & \downarrow \mathcal{L} \\ \dfrac{1}{p^2} & \to & -\left(\dfrac{1}{p^2}\right)' = \dfrac{2}{p^3} & \to & -\left(\dfrac{2}{p^3}\right)' = \dfrac{2\cdot 3}{p^4} & \to & & & -\left(\dfrac{2\cdot 3}{p^4}\right)' = \dfrac{2\cdot 3\cdot 4}{p^5} \end{array}$$

Therefore

$$\begin{array}{ccc} tu(t) & \to & t^n u(t) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ \dfrac{1}{p^2} & \to & \dfrac{2\cdot 3\cdot 4\cdots n}{p^{n+1}} = \dfrac{n!}{p^{n+1}} \end{array}$$

when $Re\, p > 0$.

10. *(Periodic function)*
Let $f(t)$ be a periodic function with fundamental period T,

$$f(t) = f(t+T), \quad t \in R.$$

Denoting by

$$\widehat{f}(t) = f(t), \quad t \in [0, T],$$

we obtain the following:

$$\begin{aligned} &\int_0^\infty f(t)e^{-pt}dt \\ = &\left[\int_0^T + \int_T^{2T} + \int_{2T}^{3T} + \cdots\right] f(t)e^{-pt}dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T f(t)e^{-pt}dt + \\
+\int_T^{2T} f(t)e^{-pt}dt &= \int_0^T f(t+T)e^{-p(t+T)}dt \\
+\int_{2T}^{3T} f(t)e^{-pt}dt &= \int_0^T f(t+2T)e^{-p(t+2T)}dt \\
+ &\quad \cdots \\
&= \int_0^T f(t)e^{-pt}\left[1+e^{-pT}+e^{-p2T}+\cdots\right]dt \\
&= \int_0^T f(t)e^{-pt}\frac{1}{1-e^{-pT}}dt \\
&= \frac{1}{1-e^{-pT}}\int_0^T f(t)e^{-pt}dt = \frac{1}{1-e^{-pT}}\int_0^\infty \widehat{f}(t)e^{-pt}dt \\
&= \frac{1}{1-e^{-pT}}\cdot \widehat{F}(p)
\end{aligned}
$$

when $|e^{-pT}| < 1$ (which means that $p_1 > 0$). We define by $\widehat{F}(p)$ the Laplace transform of the periodic part $\widehat{f}(t)$ of $f(t)$.

12. *(Fourier and Laplace transforms)*
Consider a function $f(t)$ of exponential order, t is the real variable from $(-\infty, +\infty)$.
The transform

$$\mathcal{L} : f(t) \to F(p) = \int_0^\infty f(t)e^{-pt}dt, \qquad p \in C^2, \tag{131}$$

is the *Laplace transform.*
The transform

$$\mathcal{F} : f(t) \to F(\omega) = \int_{-\infty}^\infty f(t)e^{-j\omega t}dt, \qquad \omega \in R, \tag{132}$$

is the *Fourier transform,* which can be written as

$$\mathcal{L} : f(t) \to F(j\omega) = \int_0^\infty f(t)e^{-(j\omega)t}dt, \qquad \omega \in R, \tag{133}$$

if $f(t) = 0$, for all $t < 0$.
So, for functions $f(t) = 0$, $t < 0$, the Fourier transform $F(\omega)$ is the part of the Laplace transform $F(p)$, i.e., the Laplace transform on the line $p = j\omega, \omega \in R$. It is assumed that the Laplace transform exists on the vertical line $p = j\omega$.

For example,

$$f(t) = e^{-\alpha t} u(t), \quad \alpha > 0,$$

$$\mathcal{L}\{f(t)\}(p) = F(p) = \frac{1}{p+\alpha}, \qquad p \in C^2, \quad (Re\, p > 0)$$

and

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) = \frac{1}{j\omega + \alpha} \qquad \omega \in R.$$

We have defined the Laplace transform of $f(t)$ for p such that $Re\, p > 0$, and not for $Re\, p = 0$.

$$F(p)|_{p=j\omega} = F(\omega).$$

But, for the function

$$f(t) = e^{-\alpha|t|}, \quad \alpha > 0,$$

the Fourier image is

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) = \frac{2\alpha}{\omega^2 + \alpha^2}, \qquad \omega \in R$$

but this function is not zero for negative ω, and the Laplace transform cannot be defined.

The following example, when these two transform are "not equal":

$$f(t) = u(t).$$

Indeed,

$$\mathcal{L}\{f(t)\}(p) = U(p) = \frac{1}{p}, \qquad Re\, p > 0, \tag{134}$$

and

$$\mathcal{F}\{f(t)\}(\omega) = U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}, \qquad \omega \in R.$$

In this case also, we have defined the Laplace transform of $u(t)$ for p such that $Re\, p > 0$, and formula (134) cannot be used for vertical line $Re\, p = 0$. Indeed,

$$\frac{1}{p = jw} = \frac{1}{j\omega} \neq \frac{1}{j\omega} + \pi\delta(\omega) = U(\omega).$$

So, we extend the Laplace transform on the vertical $Re\, p = 0$, defining the Laplace transform as the Fourier transform

$$U(p)|_{p=j\omega} = U(\omega).$$

13. *(Stability of causal systems)*

Let $h(t)$ be an impulse response function such that $h(t) = 0$, $t < 0$, and let $H(p)$ be the Laplace transform of $h(t)$. We consider the transfer function of an nth-order system $x(t) \to y(t)$,

$$H(p) = \frac{Y(p)}{X(p)} = \frac{b_{n-1}p^{n-1} + ... + b_1 p + b_0}{p^n + a_{n-1}p^{n-1} + ... + a_1 p + a_0}$$

where $b_{n-1} \neq 0$.

We know, that the transfer function can be represented as

$$H(p) = \frac{b_1'}{p - p_1} + \frac{b_2'}{p - p_2} + ... + \frac{b_n'}{p - p_n}$$

where $p_1, p_2, ..., p_n$ are zeros of the denominator (or, *poles* of the $H(p)$), i.e.

$$p^n + a_{n-1}p^{n-1} + ... + a_1 p + a_0 = (p - p_1)(p - p_2) \cdots (p - p_n).$$

The output transform can be represented as

$$Y(p) = \frac{k_1}{p - p_1} + \frac{k_2}{p - p_2} + ... + \frac{k_n}{p - p_n} + Y_X(p)$$

where $Y_X(p)$ has the same poles as $X(p)$.

Therefore,

$$y(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} + ... + k_n e^{p_n t} + y_x(t).$$

The system H is unstable when the output is unbounded for bounded input $x(t)$, or when there exists

$$p_m > 0, \quad m \in [1, n].$$

For example,

$$H(p) = \frac{1}{p+2} + \frac{2}{p-3} + \frac{3}{p+4} \longleftarrow e^{-2t} + \underline{2e^{3t}} + 3e^{-4t}$$

is for not stable system ($p_2 = 3 > 0$), but

$$H(p) = \frac{1}{p+2} + \frac{2}{p+3} + \frac{3}{p+4} \longleftarrow e^{-2t} + 2e^{-3t} + 3e^{-4t}$$

is the transfer function for the stable system.

We remain here, that a complex function

$$X(p) : C^2 \to C^2, \qquad (C^2 = R \times jR)$$

has a pole at point $p = p_0$ if

$$\lim_{p \to p_0} X(p) = \pm\infty,$$

and we say this function has a zero at point $p = p_0$ if

$$\lim_{p \to p_0} X(p) = 0.$$

For example, functions

$$X(p) = \frac{1}{p+2}, \quad X(p) = (p-2), \quad X(p) = \frac{p-2}{p+2}$$

have poles or/and zeros respectively at points $p = -2$ and $p = 2$.

A rational function is a ratio of two (numerator and denominator) polynomials with real coefficients

$$X(p) = \frac{P_m(p)}{P_n(p)} = \frac{b_m p^m + b_{m-1}p^{m-1} + ... + b_1 p + b_0}{p^n + a_{n-1}p^{n-1} + ... + a_1 p + a_0},$$

and $n > m$ is order of the function $X(p)$.

We consider $X(p)$ in the form

$$X(p) = \frac{P_m(p)}{p^n + a_{n-1}p^{n-1} + ... + a_1 p + a_0}$$

and assume that it has n distinct poles, $p_1, p_2, ..., p_n$.

Then, there exist coefficients $b_1, b_2, ..., b_n$ such that

$$X(p) = \frac{b_1}{p-p_1} + \frac{b_2}{p-p_2} + ... + \frac{b_n}{p-p_n}$$

(this is so-called a *partial fraction representation* of $X(p)$).

Example 9:

$$\begin{aligned} X(p) &= \frac{p+3}{p(p+2)} = \frac{p+2+1}{p(p+2)} = \frac{1}{p} + \frac{1}{p(p+2)} \\ &= \frac{1}{p} + \frac{1}{2}\left[\frac{1}{p} - \frac{1}{p+2}\right] \\ &= \frac{3}{2}\cdot\frac{1}{p} - \frac{1}{2}\cdot\frac{1}{p+2} = \frac{\frac{3}{2}}{p} - \frac{\frac{1}{2}}{p+2} \end{aligned}$$

This function has two poles at points $p_1 = 0$, and $p_2 = -2$, and $b_1 = 3/2$, and $b_2 = -1/2$. There is another simple way to find the coefficients b_1 and b_2.

Indeed,

$$pX(p)_{|p=0} = \left[\frac{p+3}{p+2}\right]_{p=0} = \frac{3}{2} = b_1,$$

$$(p+2)X(p)_{|p=-2} = \left[\frac{p+3}{p}\right]_{p=-2} = -\frac{1}{2} = b_2.$$

So, the function with the image equal $X(p)$ can be computed as

$$\begin{array}{ccccc} X(p) & = & \frac{3}{2}\cdot\frac{1}{p} & - & \frac{1}{2}\cdot\frac{1}{p+2} \\ \uparrow \mathcal{L} & & \uparrow \mathcal{L} & & \uparrow \mathcal{L} \\ x(t) & = & \frac{3}{2}\cdot u(t) & - & \frac{1}{2}\cdot e^{-2t}u(t) \end{array}$$

so

$$x(t) = \frac{3}{2} \cdot u(t) - \frac{1}{2} \cdot e^{-2t} u(t) = \frac{1}{2} \left[3 - e^{-2t}\right] u(t).$$

In the general case, to find the coefficient b_k, we calculate

$$(p - p_k) X(p)_{|p=p_k} = b_k, \qquad k = 1, 2, ..., n.$$

Example 10:

$$X(p) = \frac{1}{p^2(p+2)} = \frac{d_1}{p} + \frac{b_1}{p^2} + \frac{b_2}{p+2}.$$

We have

$$p^2 X(p)_{|p=0} = \left[\frac{1}{p+2}\right]_{p=0} = \frac{1}{2} = b_1,$$

$$(p+2) X(p)_{|p=-2} = \left[\frac{1}{p^2}\right]_{p=-2} = \frac{1}{4} = b_2,$$

therefore

$$X(p) = \frac{d_1}{p} + \frac{\frac{1}{2}}{p^2} + \frac{\frac{1}{4}}{p+2}$$

and we can find that $d_1 = -1/4$.

The function $x(t)$ with that image can be computed from the diagram:

$$\begin{array}{ccccccc} X(p) & = & -\frac{1}{4} \cdot \frac{1}{p} & + & \frac{1}{2} \cdot \frac{1}{p^2} & + & \frac{1}{4} \cdot \frac{1}{p+2} \\ \uparrow \mathcal{L} & & \uparrow \mathcal{L} & & \uparrow \mathcal{L} & & \uparrow \mathcal{L} \\ x(t) & = & -\frac{1}{4} \cdot u(t) & + & \frac{1}{2} \cdot t u(t) & + & \frac{1}{4} \cdot e^{2t} u(t) \end{array}$$

which results

$$x(t) = \left[-\frac{1}{4} + \frac{1}{2} t - \frac{1}{4} e^{2t}\right] u(t).$$

14. *Uniqueness*

Let $f(t) \neq g(t)$ be two continuous functions with an exponential order, and let their Laplace transforms are the same

$$F(p) = G(p), \qquad p \in ROC(F) \cap ROC(G).$$

Then, for any interval $[0, T]$, where $T > 0$, the functions coincide, $f(t) = g(t)$, except maybe for a finite number of points.

14. *Limits of $pF(p)$*

A. *Initial value theorem:*

$$f(0) = \lim_{p\to\infty} pF(p). \tag{135}$$

B. *Final value theorem:* If the limit $\lim_{t\to\infty} f(t)$ exists, then

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} pF(p). \tag{136}$$

The following example shows the necessity of condition for this property. Consider the function

$$F(p) = \frac{p}{p^2+4}$$

Then

$$F(p) = \frac{p}{p^2+4} \xleftarrow{\mathcal{L}} \cos(2t)u(t) \to \lim_{t\to\infty} \cos(2t)u(t) \text{ does not exist.}$$

Example 11:

$$F(p) = \frac{p+2}{p^2+4p+5}.$$

Then

$$f(0) = \lim_{p\to\infty} pF(p) = \lim_{p\to\infty} \frac{p^2+2p}{p^2+4p+5} = 1$$

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} pF(p) = \lim_{p\to 0} p \cdot \frac{p+2}{p^2+4p+5} = 0.$$

Example 12:

$$F(p) = \frac{p+2}{p^2+4p-5} = \frac{\frac{1}{2}}{p-1} + \frac{\frac{1}{2}}{p+5}$$

Then

$$f(0) = \lim_{p\to\infty} pF(p) = \lim_{p\to\infty} \left[\frac{1}{2}\frac{p}{p-1} + \frac{1}{2}\frac{p}{p+5}\right] = 1$$

$$\lim_{p\to 0} pF(p) = \lim_{p\to 0} \left[\frac{1}{2}\frac{p}{p-1} + \frac{1}{2}\frac{p}{p+5}\right] = 0 =? = \lim_{t\to\infty} f(t).$$

The property of FVT cannot be applied. Indeed

$$F(p) = \frac{1}{2}\frac{1}{p-1} + \frac{1}{2}\frac{1}{p+5} \to f(t) = \frac{1}{2}e^{t}u(t) + \frac{1}{2}e^{-5t}u(t)$$

and the first component of $f(t)$ does not have a finite limit when t tends to the positive infinity.

15. *Using MATLAB*
Consider the following function as the Laplace transform

$$F(p) = \frac{p+2}{p^3+2p^2+4p+5}.$$

The following next show a way to calculate perform the partial fraction expansion in MATLAB.

(1) Define two vectors with coefficients of numerator and denominator
≫ num=[1 2]; den=[1 2 4 5];

(2) Use "residue" function to finds the residues, poles, and direct term of the partial fraction expansion of $F(p)$.
≫ [r,p,k]=residue(num,den);

As a result, we obtain the following data:

r =
–0.0486 – 0.3135i
–0.0486 + 0.3135i
0.0971

p =
–0.2370 + 1.7946i
–0.2370 – 1.7946i
-1.5260

k =
[ ]

(3) Therefore, the Laplace transform can be written as follows

$$F(p) = \frac{-0.0486-0.3135i}{p-(-0.2370+1.7946i)} + \frac{0.0486+0.3135i}{p-(-0.2370-1.7946i)} + \frac{0.0971}{p-(-1.5260)}$$

or

$$F(p) = \frac{-0.0486-0.3135i}{p+0.2370-1.7946i} + \frac{0.0486+0.3135i}{p+0.2370+1.7946i} + \frac{0.0971}{p+1.5260}.$$

16. *Laplace transform in Circuit Analysis*
<u>Resistance R</u>:

$$v_R(t) = Ri_R(t) \xrightarrow{\mathcal{L}} V_R(p) = RI_R(p).$$

<u>Inductor L</u>:

$$v_L(t) = L\frac{di_L(t)}{dt} \xrightarrow{\mathcal{L}} V_L(p) = L\left[I_L(p) - i_L(0)\right].$$

<u>Capacitor C</u>:

$$v_C(t) = \frac{1}{C}\int_0^t i_C(\tau)d\tau + v_C(0) \xrightarrow{\mathcal{L}} V_C(p) = \frac{1}{C}\cdot\frac{1}{p}I_C(p) + \frac{v_C(0)}{p}.$$

17. *Forced and Natural Responses*

We consider a linear system described by the following differential equation

$$y''(t) + 7y'(t) + 12y(t) = x'(t) + 3x(t) + \underline{0}, \qquad t > 0 \tag{137}$$

with the initial conditions $y'(0) = -4$, $y(0) = 2$.

We now find the response of the system to the input $x(t) = u(t)$. For that, we recall that we have

$$\begin{aligned}
&y''(t) \xrightarrow{\mathcal{L}} p^2Y(p) - py(0) - y'(0) = p^2Y(p) - 2p + 4\\
&y'(t) \xrightarrow{\mathcal{L}} pY(p) - y(0) = pY(p) - 2\\
&x'(t) \xrightarrow{\mathcal{L}} pX(p) - x(0) = pX(p) - \begin{cases} 1, & \text{if} \quad u(0) = 1\\ 0.5, & \text{if} \quad u(0) = 0.5\end{cases}
\end{aligned}$$

and we consider that $u(0) = 1$, then $pX(p) - 1 = pU(p) - 1 = 0$ for all p such that $Rep > 0$.

Applying the Laplace transform over both parts of the differential equation in (137), we obtain

$$\begin{aligned}
\left[p^2Y(p) - 2p + 4\right] + 7\left[pY(p) - 2\right] + 12Y(p) &= 3X(p)\\
Y(p)[p^2 + 7p + 12] - 2p - 10 &= 3\frac{1}{p}, \quad Rep > 0
\end{aligned}$$

$$Y(p) = \underbrace{\frac{2p + 10}{p^2 + 7p + 12}}_{NR} + \underbrace{\frac{3}{p(p^2 + 7p + 12)}}_{FR}.$$

The Laplace transform is composed from the transforms of the natural response (NR) and forced response (FR) $Y(p) = Y_N(p) + Y_F(p)$, where

$$\begin{aligned}
Y_N(p) &= \frac{2p + 10}{p^2 + 7p + 12} = \frac{A_1}{p + 3} + \frac{B_1}{p + 4}\\
Y_F(p) &= \frac{3\frac{1}{p}}{p^2 + 7p + 12} = \frac{A_2}{p + 3} + \frac{B_2}{p + 4} + \frac{C_2}{p}
\end{aligned}$$

where it is not difficult to find that $A_1 = 4$, $B_1 = -2$, and $A_2 = -1$, $B_2 = 3/4$, $C_2 = 1/4$. The Laplace transform of the natural response have two poles -3 and -4, and the Laplace transform of the forced response has one pole 0 more.

We now consider the inverse Laplace transforms of $Y_N(p)$ and $Y_F(p)$:

$$\begin{aligned}
Y_N(p) &= \frac{4}{p + 3} + \frac{-2}{p + 4} \xrightarrow{\mathcal{L}^{-1}} y_N(t) = 4e^{-3t}u(t) - 2e^{-4t}u(t)\\
Y_F(p) &= \frac{-1}{p + 3} + \frac{3}{4} \cdot \frac{1}{p + 4} + \frac{1}{4} \cdot \frac{1}{p} \xrightarrow{\mathcal{L}^{-1}} y_F(t) = -e^{-3t}u(t) + \frac{3}{4}e^{-4t}u(t) + \frac{1}{4}u(t).
\end{aligned}$$

The response of the system (as the inverse Laplace transform of $Y(p)$) to the input $u(t)$ equals

$$\begin{aligned}
y(t) &= y_N(t) + y_F(t)\\
&= 4e^{-3t}u(t) - 2e^{-4t}u(t) - e^{-3t}u(t) + \frac{3}{4}e^{-4t}u(t) + \frac{1}{4}u(t)\\
&= \left[3e^{-3t} - \frac{5}{4}e^{-4t} + \frac{1}{4}\right]u(t).
\end{aligned}$$

We can check the initial conditions for the obtained response

$$y(0) = 3 - \frac{5}{4} + \frac{1}{4} = 2, \qquad y'(0) = (-3)3 - (-4)\frac{5}{4} = -9 + 5 = -4.$$

Bilateral Laplace Transform

$$F(p) = \int_{-\infty}^{\infty} f(t)e^{-pt}dt, \quad p \in C^2.$$

For the function of exponential order the following holds

$$f(t) = Ae^{\alpha t}u(t) \xrightarrow{\mathcal{L}} \frac{A}{p-\alpha}, \quad Rep > \alpha.$$

We now consider the time-inverted function

$$g(t) = f(-t) = Ae^{-\alpha t}u(-t).$$

The Laplace transform of $g(t)$ is calculated as follows

$$\begin{aligned} G(p) &= \int_{-\infty}^{\infty} f(-t)e^{-pt}dt = -\int_{+\infty}^{-\infty} f(-t)e^{-(-p)(-t)}d(-t) \\ &= \int_{-\infty}^{\infty} f(t)e^{-(-p)t}dt = F(-p) = -\frac{A}{p+\alpha} \end{aligned}$$

for any p such that $Re(-p) > \alpha \Rightarrow Rep < -\alpha$.

Thus we have

$$\boxed{\begin{array}{cc} Ae^{\alpha t}u(t) & Ae^{-\alpha t}u(-t) \\ \downarrow \mathcal{L} & \downarrow \mathcal{L} \\ \dfrac{A}{p-\alpha} & -\dfrac{A}{p+\alpha} \\ \\ Re\,p > \alpha & Re\,p < -\alpha \end{array}}$$

This example illustrates also the importance of the ROC for the Laplace transform. Indeed, the following diagram holds

$$\boxed{\begin{array}{ccc} Ae^{-\alpha t}u(t) & & -Ae^{-\alpha t}u(-t) \\ \searrow \mathcal{L} & & \swarrow \mathcal{L} \\ & \dfrac{A}{p+\alpha} & \\ Re\,p > -\alpha & & Re\,p < -\alpha \end{array}}$$

Thus for the Laplace transform, the ROC should be specified.

Properties of the bilateral Laplace transform are similar to unilateral Laplace transform, except the following two.

1. *(Time scaling)*

For any real number $k \neq 0$,

$$\begin{array}{ccc} f(t) & \to & g(t) = f(kt) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ F(p) & \to & G(p) = \dfrac{1}{|k|} X(\dfrac{p}{k}) \end{array}$$

and ROC(G) =ROC(F/k).

2. *(Derivative)*

$$\begin{array}{ccc} f(t) & \to & f'(t) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ F(p) & \to & pF(p) \end{array} \qquad (Rep > a)$$

Consider the following non-causal signal

$$x(t) = 3e^{-4t}u(t) + 2e^{-2t}u(-t)$$

and the impulse response function

$$h(t) = 2\delta(t) - e^{-3|t|}$$

of a non-causal system.

The following calculations hold for the Laplace transform

$$\boxed{\begin{array}{ccc} 3e^{-4t}u(t) & + & 2e^{-2t}u(-t) \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ \dfrac{3}{p+4} & + & -\dfrac{1}{p+2} \\ \\ \{Re\, p > -4\} & \cap & \{Re\, p < -2\} \end{array}}$$

and ROC(X)=$\{p;\ -4 < Re\, p < -2\}$.

For the impulse response

$$\boxed{\begin{array}{ccc} 2\delta(t) & - & e^{-3|t|} \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ 2 & - & \dfrac{-2 \cdot 3}{p^2 - 9} \\ \\ \forall p & \cap & \{|Re\, p| < 3\} \end{array}}$$

and $ROC(H) = \{p;\ -3 < Re\ < 3\}$.

We can define the Laplace transform of the convolution in the region

$$\text{ROC}(Y) = \{p;\ -3 < Re\, p < -2\},$$

where

$$\begin{aligned} Y(p) &= H(p)X(p) = \left[\frac{3}{p+4} - \frac{2}{p+2}\right] \cdot \left[2 + \frac{6}{p^2-9}\right] \\ &= \frac{A_1}{p+4} + \frac{A_2}{p+2} + \frac{A_3}{p-3} + \frac{A_4}{p+3} \end{aligned}$$

for some numbers $A_1, ..., A_4$.

XIX. Brief Notes in Sampling Theorem *EE-3424*

I. Digital signal processing provides an alternative method for processing the analog signals. To perform processing digitally, there is a need for an interface between the analog signal and digital processor. This interface is called an *analog-to-digital* (A-D) converter. The output of A-D is a digital signal to be processed in the digital processor.

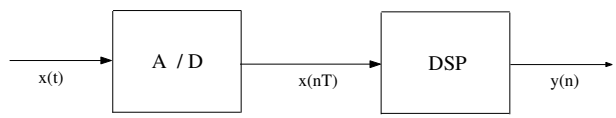

Fig. 89. Diagram of digital signal processing.

We consider the sampling process, where the analog signal $x(t)$ (or continuous-time signal) is measured periodically every T seconds, as $x(nT)$. T is the *sampling time*, or *period*, and time is counted at points multiple to T, i.e. $t = 0, T, 2T, 3T, \dots,$ ($t = nT$, n=0,...), or $t = 0, \pm T, \pm 2T, \pm 3T, \dots$.

The following problems are considered:

1. How does the sampling effect on the Fourier spectrum of the original signal?
2. What is the maximal value of sampling interval T?
3. Is it possible to reconstruct the original continuous-time signal $x(t)$ from its sampled version, by interpolating between the samples $x(nT)$.

We consider an absolute (or square) integrable function $x(t)$ defined on the real line R for which the *Fourier transform* exists

$$X(\omega) = (\mathcal{F} \circ x)(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t}\, dt, \quad \omega \in (-\infty, +\infty). \tag{138}$$

The *inverse* Fourier transform is described as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t}\, d\omega, \quad t \in (-\infty, +\infty). \tag{139}$$

A.

Let T be an interval of sampling the function $x(t)$, that results in the sequence

$$x(t) \to \boxed{\text{Analog/Digital Converter}} \to x(nT)$$

$$x(nT) = x(t)_{|t=nT}, \quad n = 0, \pm 1, \pm 2, \dots .$$

We consider the relation between the spectrum of the sequence $x(nT)$ with the Fourier transform of the continuous-time representation of this sequence, or sampled signal,

$$x_s(t) = x(t)s(t) = x(t) \sum_{n=0,\pm 1,\pm 2,\dots} \delta(t - nT) = \sum_{n=0,\pm 1,\pm 2,\dots} x(nT)\delta(t - nT)$$

We need to remember that the train of delta functions is represented by the train of delta functions in the frequency domain as

$$\sum_{n=0,\pm1,\pm2,\ldots} \delta(t-nT) \to \omega_0 \sum_{n=0,\pm1,\pm2,\ldots} \delta(\omega - n\omega_0)$$

where $\omega_0 = 2\pi/T$. Therefore, in the frequency domain, this equation can be written as (see more details in pages 106-107)

$$X_s(\omega) = \frac{1}{T} \sum_{n=0,\pm1,\pm2,\ldots} X(\omega - n\frac{2\pi}{T}) = \frac{1}{T} \sum_{n=0,\pm1,\pm2,\ldots} X(\omega - n2\pi f_s). \tag{140}$$

The sampling process generates high frequency components and every frequency component of the original signal is periodically replicated over entire frequency axis with period

$$f_s = \frac{1}{T} \qquad (\textit{sampling rate (frequency)})$$

in units of samples per second. [1]

B. Let us assume that the signal has bounded spectrum, i.e. $X(\omega) = 0$ for all $|\omega| > \Omega$, and $\pi/T > \Omega$ for a positive number Ω. In this case, for all integers $m \neq 0$, we have

$$X\left(\omega + m\frac{2\pi}{T}\right)\Big|_{(-\Omega,\Omega)} = 0.$$

This means that if a frequency $\omega \in (-\Omega, \Omega)$, then $X\left(\omega + m\frac{2\pi}{T}\right) = 0$ and the replicas of $X(\omega)$ do not overlap (see Fig. 90(a)).

Then, we can state that the Fourier transform of the continuous-time signal can be defined from the discrete Fourier transform of the sampled signal as

$$\frac{1}{T}X(\omega) = X_s(\omega), \quad \text{or} \quad X(\omega) = TX_s(\omega) \tag{141}$$

for all frequencies $|\omega| \leq \Omega$.

And opposite, if period of sampling T such that $\pi/T < \Omega$, then at least two additional terms $X\left(\omega + \frac{2\pi}{T}\right)$ and $X\left(\omega - \frac{2\pi}{T}\right)$ will contribute to form the spectrum of the sampled signal in the interval $[-\Omega, \Omega]$ as shown in Fig. 90(b), and the Fourier transform $X(\omega)$ in the interval $[-\Omega, \Omega]$ cannot be defined as

$$X(\omega) = TX_s(\omega), \quad \omega \in [-\Omega, \Omega],$$

from the spectrum of the sampled signal. The reason is the small sampling rate f_s (or, not small sampling time T), which results in the overlapping (aliasing) condition

$$\frac{\pi}{T} < \Omega.$$

[1] The periodicity is in term of Hz and means that $X_s(\omega) = X_s(2\pi f)$ is periodic when shifting $f \to f + f_s$.

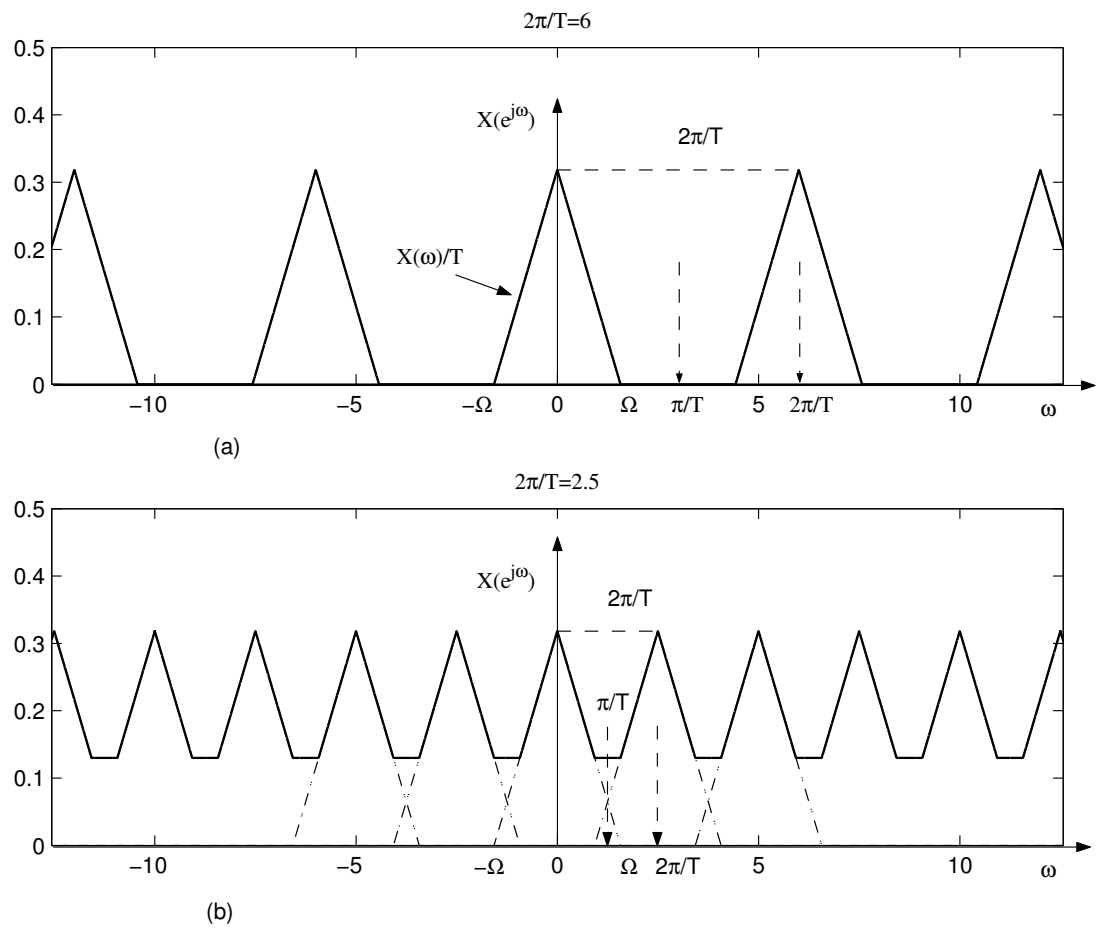

Fig. 90. (a) Effect of sampling when $\pi/T > \Omega$, and (b) effect of sampling when $\pi/T < \Omega$.

To determine the Fourier transform of the function $x(t)$, the period of sampling should be taken as a period T_1 which provides the condition

$$-\frac{\pi}{T_1} < -\Omega \quad \text{and} \quad \Omega < \frac{\pi}{T_1}$$

and, therefore $T_1 \leq T$.

C. From the first example of Fig. 90 (part a), one can see that it is possible to reconstruct the original spectrum from the continuous-discrete Fourier transform, taking

$$X(\omega) = TX_s(\omega), \qquad |\omega| \leq \frac{\pi}{T} > \Omega \tag{142}$$

since $X(\omega) \equiv 0$ when ω lies outside the interval $(-\pi/T, \pi/T)$. From the second example, we observe that one can never reconstruct the original spectrum from $X(e^{j\omega T})$. Namely, the original Fourier transform can be reconstructed partially (reconstruction for low frequencies).

It should be noted that the sampling condition in terms of frequencies in Hz is derived as follows

$$\frac{\pi}{T} \geq \Omega \quad \Leftrightarrow \quad \frac{1}{2T} = \underline{\frac{f_s}{2} \geq f_b} = \frac{\Omega}{2\pi}.$$

(see illustration of this condition in Figure 91). In other words, the sampling rate $f_s \geq 2f_b$.

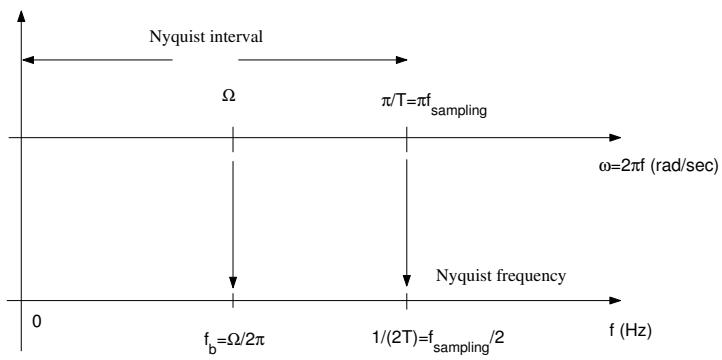

Fig. 91. Condition of the proper sampling the signal bounded by Ω.

D. The following statement holds.

Theorem 1: Let $x(t)$ be a function with the bounded spectrum, i.e.

$$X(\omega) = 0, \qquad |\omega| > \Omega \tag{143}$$

and let such value $\Omega > 0$ exists. Then $x(t)$ can be described uniquely by its samples taken at discrete time with the sampling interval

$$T < \frac{\pi}{\omega_s}$$

for a frequency ω_s such that $\Omega \leq \omega_s \leq \pi/T$.

This statement has been proved first by Kotelnikov [V.A. Kotelnikov, *Theory of potential noise stability,* Moscow, Nauka, 1956] but is often called Uttekker and Shannon's sampling theorem.

Example 1: Consider the signal $x(t)$ that is bounded in the spectral domain of $50 \cdot 10^3$rad/sec. Then, $\Omega = 25 \cdot 10^3$rad/sec. Let us assume that $T = 10^{-4}$sec. Checking the condition

$$\omega_{period/2} = \frac{\pi}{T} = 3.14 \cdot 10^4 > \Omega$$

we see that T can be considered as a good sampling period.

If we take $T = 1.5 \cdot 10^{-4}$sec, then after checking the condition

$$\omega_{period/2} = \frac{\pi}{T} = 3.14/1.5 \cdot 10^4 \approx 2.28 \cdot 10^4 < \Omega$$

we can state that such T cannot be used for sampling a signal with bounded spectrum of $50 \cdot 10^3$rad/sec. We need reduce by ΔT the sampling period for signal reconstruction. The minimum change in sampling time is defined as

$$\Delta T = T - \frac{\pi}{\Omega} = 1.5 \cdot 10^{-4} - \frac{\pi}{2.5 \cdot 10^4} = 10^{-4}(1.5 - \frac{\pi}{2.5}) = 2.4336 \cdot 10^{-5}\text{sec}.$$

In terms of hertz, the spectrum is bounded by the frequency

$$f_b = \frac{\Omega}{2\pi} = \frac{25 \cdot 10^3}{2\pi}\text{Hz} = 3.9789 \cdot 10^3\text{Hz}$$

the sampling rate f_s therefore should be greater or equal $2f_b = 7.9578 \cdot 10^3$Hz.

Owing to the Kotelnikov theorem (given bellow), each function with the bounded spectrum can be restored from its discrete values chosen in the defined way. So, the one-dimensional signal $x(t)$, that is given in the finite interval $[0, L]$ and satisfies the condition of spectrum to be band limited: $X(\omega) = 0$ for all $|\omega| > \Omega$, can be represented one-to-one by the discrete sequence of its values

$$\{x_n = x(nT), \quad n = 0 : (N-1)\} \tag{144}$$

taken with a sampling interval $T \leq \pi/\Omega = 1/(2f_b)$ such that $N = L/T$ is an integer. The restoration of the assumed continuous-time function from its discrete values (144) is described by the expansion formula of the function:

$$x(t) = \sum_{n=0}^{N-1} x_n \text{sinc}\{\Omega(t - nT)\}, \quad t \in [0, L] \tag{145}$$

by the basis functions

$$sinc\{\Omega(t - nT)\} = \frac{\sin\{\Omega(t - nT)\}}{\Omega(t - nT)} \tag{146}$$

that are the shifted by nT and time-scaled by Ω versions of $sinc(t)$ functions

$$sinc(t) \to sinc(\underline{\Omega}t) \to \text{sinc}(\Omega(t - \underline{nT})).$$

Thus, a "signal" with the bounded spectrum (143) is defined completely by the finite number of its values at points situated uniformity on the segment $[0, L]$.

In the proof the statement of the Kotelnikov theorem in (145), the condition of not aliasing is used,

$$X(\omega) = TX_s(\omega), \qquad |\omega| < \Omega < \frac{\pi}{T}. \tag{147}$$

In the general case, Eq. 145 has the form

$$\begin{aligned} x(t) &= \frac{T\Omega}{\pi} \sum_{n \in Z} x(nT) \text{sinc}\left[\Omega(t - nT)\right] \\ &= \sum_{n \in Z} x(nT) \text{sinc}\left[\Omega(t - nT)\right] \qquad (\text{ if } \frac{\pi}{T} = \Omega). \end{aligned}$$

E. In the real word, the signals are not bound limited. They are limited in time but not in the frequency domain.

Example 2: Consider the rectangle function on the interval $[-T_0, T_0]$

$$x(t) = rect\left(\frac{t}{2T_0}\right) = u(t + T_0) - u(t - T_0) \to 2T_0 \,\text{sinc}(\omega T_0).$$

In the particular $T_0 = 1/2$ case, we obtain $rect(t) \xrightarrow{\mathcal{F}} \text{sinc}(\omega/2)$. Note that $\text{sinc}(\omega)$ is the function that is not limited neither absolute integrable.

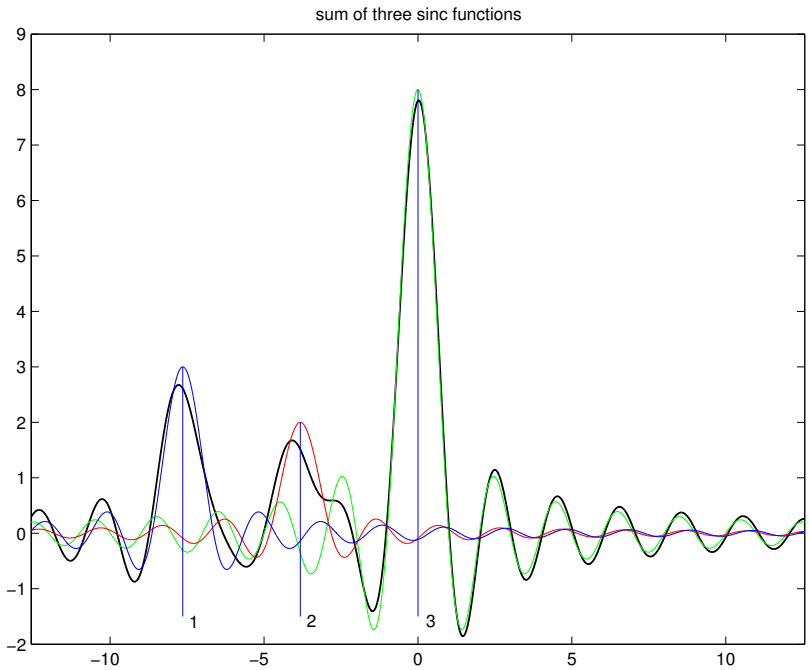

Fig. 92. Elements of the decomposition of the reconstructed signal by basis $sinc(t)$ functions.

D. We consider a continuous-in-time signal that itself represents a step-function with values of the sampled sequence

$$\widetilde{x}(t) = x(nT), \qquad t \in A_n = (nT - T/2, nT + T/2], \quad n \in Z. \tag{148}$$

The function $\widetilde{x}(t)$ can be represented as the infinite sum of the pulse signals

$$\widetilde{x}(t) = \sum_{n \in Z} x(nT) rect\left(\frac{t}{T} - n\right).$$

Each function $r(t) = rect\left(\frac{t}{T} - n\right)$ represents the performance of the following time-transformations $t :\rightarrow t/T$ and $t \rightarrow t - nT$ of the $rect(t)$ function. Therefore, the Fourier transform is defined by these transformation as it shown in the diagram:

$$\begin{array}{ccccccccc}
rect(t) & \rightarrow & z(t) & = & rect\left(\frac{t}{T}\right) & \rightarrow & z(t-nT) & = & rect\left(\frac{t}{T} - n\right) \\
\downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} & = & \downarrow \mathcal{F} \\
\text{sinc}\left(\frac{\omega}{2}\right) & \rightarrow & Z(\omega) & = & T\text{sinc}\left(\frac{\omega T}{2}\right) & \rightarrow & Z(\omega)e^{-j\omega nT} & = & T\text{sinc}\left(\frac{\omega T}{2}\right)e^{-j\omega nT}
\end{array}$$

Therefore, the Fourier transform of the function $\widetilde{x}(t)$ is calculated as

$$\begin{aligned}
\widetilde{X}(\omega) &= \sum_{n \in Z} x(nT) T\text{sinc}\left(\frac{\omega T}{2}\right) e^{-j\omega nT} \\
&= \sum_{n \in Z} x(nT) e^{-j\omega nT} \cdot \left[T\text{sinc}\left(\frac{\omega T}{2}\right)\right] = T X_s(\omega)\text{sinc}\left(\frac{\omega T}{2}\right).
\end{aligned}$$

On the other hand, if $x(nT)$ is the sampled signal $x(t)$ with the sampling period T, then according to (140)

$$X_s(\omega) = \frac{1}{T}X(\omega) + \frac{1}{T}\sum_{m=\pm1,\pm2,\ldots} X\left(\omega + m\frac{2\pi}{T}\right).$$

Therefore,

$$\begin{aligned}\widetilde{X}(\omega) &= TX_s(\omega)\mathrm{sinc}\left(\frac{\omega T}{2}\right)\\ &= X(\omega)\,\mathrm{sinc}\left(\frac{\omega T}{2}\right) + \left[\sum_{m\neq 0} X\left(\omega + m\frac{2\pi}{T}\right)\right]\mathrm{sinc}\left(\frac{\omega T}{2}\right).\end{aligned}$$

and we can see that in general

Even in the case when $x(t)$ has a spectrum bounded by $\Omega < \pi/T$, the following holds for $\omega \in (-\Omega, \Omega)$

$$\widetilde{X}(\omega) = X(\omega)\mathrm{sinc}\left(\frac{\omega T}{2}\right) \neq X(\omega) \qquad (\text{if } \omega \neq 0)$$

which shows the error of the digital-to-analog convertor.

F. (*Periodic sampling step function*)

We consider the mathematical representation of the sampling

$$x(t) \to x_s(t) = x(t)\cdot s(t) = \begin{cases} x(nT), & \text{if } t = nT \\ 0, & \text{otherwise}\end{cases}$$

when the continuous-time signal is transformed into a continuous-time signal with values of the original signal at points nT. The modulated signal $s(t)$ is periodic impulse function with period T

$$\begin{aligned}s(t) &= \{\ldots, \delta(t+2T), \delta(t+T), \delta(t), \delta(t-T), \delta(t-2T), \ldots\}\\ &= \sum_{n\in Z}\delta(t-nT) = \frac{1}{T}\sum_{n\in Z} e^{-jn\omega_s t}\end{aligned}$$

where $\omega_s = 2\pi/T$ is the sampling frequency.

In the Fourier domain, we following diagram holds for the function $s(t)$

$$\begin{array}{ccc} s(t) & = & \frac{1}{T}\sum_{n\in Z} e^{-jn\omega_s t} \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ S(\omega) & = & \frac{1}{T}\sum_{n\in Z} 2\pi\delta(\omega - n\omega_s)\end{array}$$

Next, because of the property of the convolution in the frequency domain, we obtain the following

$$\begin{array}{ccc} x_s(t) & = & x(t)\cdot s(t) \\ \downarrow \mathcal{F} & = & \downarrow \mathcal{F} \\ X_s(\omega) & = & \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega-\tau)S(\tau)d\tau\end{array}$$

and

$$\begin{aligned} X_s(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega-\tau)\left[\frac{1}{T}\sum_{n\in Z} 2\pi\delta(\tau-n\omega_s)\right]d\tau = \frac{1}{T}\sum_{n\in Z}\int_{-\infty}^{\infty} X(\omega-\tau)\delta(\tau-n\omega_s)d\tau \\ &= \frac{1}{T}\sum_{n\in Z} X(\omega-n\omega_s) = \frac{1}{T}X(\omega) + \frac{1}{T}\sum_{n\in Z\setminus\{0\}} X(\omega-n\omega_s). \end{aligned}$$

We can see again that the Fourier transform of the function $x_s(t)$ consists of periodic copies (withe period ω_s) of the scaled Fourier transform of the original signal $x(t)$. If the Fourier transform of $x(t)$ is bound limited, $X(\omega)=0$ for $\omega \notin (-\Omega,\Omega)$, then these copies do not overlap if $\omega_s > 2\Omega$. In this case, we have

$$X(\omega) = TX_s(\omega), \quad \forall \omega \in (-\Omega,\Omega).$$

In other words, the "lowpass" filter

$$H_{lp}(\omega) = \begin{cases} T, & \text{if } |\omega| < \omega_{cut} \\ 0, & \text{otherwise} \end{cases} \quad \xrightarrow{\mathcal{F}^{-1}} \quad h_{lp}(t) = \frac{T\omega_{cut}}{\pi} sinc(\omega_{cut}t)$$

with a cutoff frequency $\omega_{cut} = \omega_{cutoff}$ from the interval $(\Omega, \omega_s/2)$ can be used to recover the Fourier transform

$$X(\omega) = H_{lp}(\omega)X_s(\omega).$$

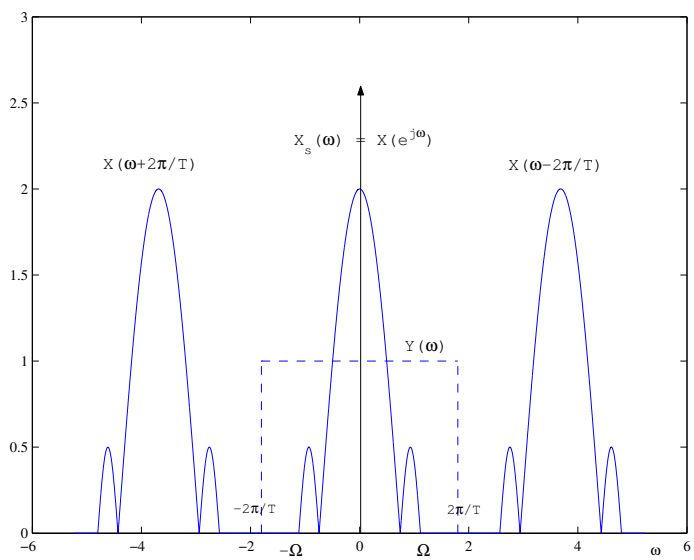

Fig. 93. Fourier transform calculation by the lowpass filter $H(\omega)$.

In the case when $\omega_s < 2\Omega$ (or $T > \pi/\Omega$), the Fourier transform of the signal $x(t)$ cannot be recovered from the Fourier transform of the signal $x_s(t)$, because of overlapping of copies $X(\omega - n\omega_s)$. Thus, $X(\omega) \neq TX_s(\omega)$, or $X(\omega) \neq H_{lp}(\omega)X_s(\omega)$ for any lowpass filter.

MATLAB-Based Project: The Fourier series and convolution of signals
Fast Fourier Series - or - the N-point Discrete Fourier Transform [1]

Time: April 6 - April 29 (by 5:00pm)

Instructor: Dr. Artyom Grigoryan

Student name:

Section:

[1]For this project you will be given maximum of 10 points which will be added to your final score.

Speech signal and linear convolution

The speech signal for this project was posted on our web page in BB. Write your code[2] to calculate the convolution of the speech signal by performing the following steps.

1. Read the speech signal $\{f_n\}$ from the file 'mike.wav' (see Figure 1); for that the command "wavread" can be used. (To sound this signal f, use the command "sound(f,Fs)". This signal is of 38.7360 seconds

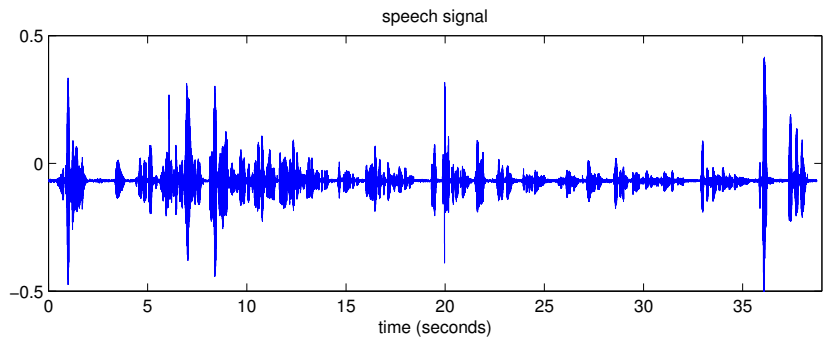

Fig. 1. Original 1-D signal.

duration.

2. Consider the triangle filter

$$\{h_n\} = \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, \underline{8}, 7, 7, 6, 6, 5, 5, 4, 4, 3, 3, 2, 2\}/K$$

with the center $h_0 = 8$; note the filter is not symmetric. Define value of K from the condition

$$\sum_{n=-12}^{12} h_n = 1.$$

3. Calculate and sketch the convolution of the signal with this filter.

4. Divide the signal f_n by parts of 7 seconds each; add zeros to the last part to complete it to 7 seconds.

5. Perform the linear convolution of each part of the signal by this filter, by using the direct method of convolution.

6. Compose one new signal from these convoluted parts; this signal, g_n, should be of the same length as the original speech signal.

7. Repeat steps 3-6 by calculating the convolutions in steps 3 and 5 by the method of the Fourier series by using the MATLAB code of the fast Fourier transform (fft).

Provide your code(s) and report for this project in hard copy with illustrations, as well as in electronic form (in one zip-file). Functions from MATLAB which calculate the convolution and filters cannot be used, only "fft" and "ifft" and "fftshift' can be used for the N-point DFT (Fourier series). Plot the signals, convolutions, Fourier transforms of the signals and filter in absolute scale. Explain clearly your work in the project, and write how fast your program(s) work.

A.M.G. (e-mail: amgrigoryan@utsa.edu)

[2]No MATLAB commands "conv", "filter",... can be used.