Algorithms of the $q^{2r} \times q^{2r}$-point 2-D Discrete Fourier Transform

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Abstract

Two methods of calculation of the 2-D DFT are analyzed.

• The $q^2 \times q^2$-point 2-D DFT can be calculated by the traditional column-row method with $2(q^2)$ 1-D DFTs, and we also propose the fast algorithm which splits each 1-D DFT by the short transforms by means of the fast paired transforms.

• The $q^2 \times q^2$-point 2-D DFT can be calculated by the tensor or paired representations of the image, when the image is represented as a set of 1-D signals which define the 2-D transform in the different subsets of frequency-points and they all together cover the complete set of frequencies. In this case, the splittings of the $q^2 \times q^2$-point 2-D DFT are performed by the 2-D discrete tensor or paired transforms, respectively, which lead to the calculation with a minimum number of 1-D DFTs.
In work, we use the concept of partitions revealing transforms for computing the 2-D DFT of order $q^2 \times q^2$, where $r > 1$ and $q$ is a positive odd number.

By means of such partitions, the 2-D discrete Fourier transform can be split into a number of short transforms, or 1-D $M$-point DFTs where $M \leq q^2$.

In the 1-D case, the partitions determine fast transformations that split the $q^2$-point DFT into a set of $N_k$-point transforms, where $k = 1:n$ and $N_1 + \ldots + N_n = q^2$, and minimizes the computational complexity of the $q^2$-point DFT.

In matrix form, the splitting can be written as

$$[\mathcal{F}_{q^2}] = \text{diag} \left\{ \begin{bmatrix} \mathcal{F}_{N_1} \\ \mathcal{F}_{N_2} \\ \vdots \\ \mathcal{F}_{N_n} \end{bmatrix} \right\} [\tilde{W}] [\chi'_{q^2}]$$

where $[W]$ is a diagonal matrix with twiddle coefficients.

We name these splitting transformations be paired $\chi'_{q^2}$. 

2D Discrete tensor and paired transforms

- 2-D DFT of the image $f_{n,m}$ of size $N \times N = q^r \times q^r$ is

$$F_{p,s} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f_{n,m} W^{np+ms}, \quad p, s = 0: (N - 1).$$

The kernel of this complex transformation $W = W_N = \exp(-2\pi j/N)$.

1. The $q^r \times q^r$-point 2-D DFT can be calculated by the column-row method with $2(q^r)$ 1-D DFTs, each of which can be
   - split by the short transforms, by means of the 1-D paired transforms
   - calculated by the scaled DFT proposed in [1] can also be used for calculating the $q^r$-point DFT, when arithmetic complexity can be reduced to $(2N-4r)$ real multiplications


2. Another and more effective algorithm of calculation of the $q^r \times q^r$-point 2-D DFT is based on the splitting by the 2-D tensor or paired transform which leads to the calculation with a minimum number of 1-D DFTs.
Tensor Representation of the \((N \times N)\) Image

The tensor representation of an image \(f_{n,m}\) which is the (2-D)-frequency-and-(1-D)-time representation, the image is described by a set of 1-D splitting-signals of length \(N\) each

\[
\chi : \{f_{n,m}\} \rightarrow \left\{ f_{T_p,s} = \{f_{p,s,t}; t = 0 : (N - 1)\} \right\}_{(p,s) \in J_{N,N}}.
\]

The components of the signals are the ray-sums of the image along the parallel lines

\[
f_{p,s,t} = \sum_{(n,m) \in X} \{f_{n,m}; np + ms = t \mod N\}.
\]

Each splitting-signals defines 2-D DFT at \(N\) frequency-points of the set

\[
T_{p,s} = \{(kp \mod N, ks \mod N); k = 0 : (N - 1)\}
\]

on the cartesian lattice \(X = \hat{X}_{N,N} = \{(n,m); n, m = 0, 1, ..., (N - 1)\}\)

\[
F_{kp \mod N, ks \mod N} = \sum_{t=0}^{N-1} f_{p,s,t} W_N^{kt}, \quad k = 0 : (N - 1).
\]
Example: 768×768-point 2-D DFT

Figure 1. (a) The image, (b) splitting-signal for (1,7), (c) the magnitude of the shifted to the middle 1-D DFT of the signal, and (d) the 2-D DFT of the image with 768 frequency-points of \( T_{1,7} \).

Figure 2. (a) The splitting-signal for (1,3), (b) 1-D DFT of the signal, and (c) the 2-D DFT of the image with the frequency-points of the sets \( T_{1,3} \) and \( T_{1,7} \).
Set of generators (p,s) in TR of Images

- Let the set $J_{N,N}$ of frequency-points (p,s), or generators, of the splitting-signals is selected in a way that covers the Cartesian lattice $X_{N,N} = \{(p,s); p,s = 0:(N−1)\}$ with a minimum number of subsets $T_{p,s}$. In other words, an irreducible covering of the Cartesian lattice is used for a certain set of generators $J_{N,N}$ in $X_{N,N}$.

$$\sigma = \sigma_{N,N} = \left( T_{p,s} \right)_{(p,s) \in J_{N,N}}$$

Example: $N=20=5(2^2)$ when $q=5$ and $r=2$, Figure 3 shows the incomplete covering of the lattice $X_{20,20}$ by 21 sets $T_{p,s}$ in part a.

Figure 3. The set of 21 subsets of the covering of the lattice 24.
Set of generators (p,s) in TR of Images

- Case 1: \(q=1\) when \(N=2^r\). The set of generators contains \(3N/2\) elements and can be defined as

\[ J_{N,N} = \{(1,s); s = 0 : (N - 1)\} \cup \{(2p,1); p = 0 : (N/2 - 1)\} . \]

- Case 2: \(q\) is prime and \(r=0\), \(N=q2^r=q\). The set of generators can be defined as

\[ J_{N,N} = \{(1,s); s = 0 : (N - 1)\} \cup \{(0,1)\} . \]

- General case: \(q\) is prime and \(r\geq 0\), \(N=q2^r\).

The irreducible covering (Tps) of the Cartesian lattice \(X_{N,N}\) has the cardinality

\[ c(N) = card \sigma_{N,N} = 2N - \varphi(N) + \sum_{p \in B_N} \beta(p). \]

\[ J_{N,N} = \bigcup_{p_2=0}^{N-1} (1,p_2) \bigcup \left( \bigcup_{p_1 \in B_N \cup \{0\}} (p_1,1) \right) \cap \left( \bigcup_{g.c.d.(p_1,p_2)=1, p_1p_2 \leq N} \{(p_1,p_2); p_1, p_2 \in B_N\} \right) . \]

Here we denote by \(\varphi(N)\) the Euler function, i.e., the number of the positive integers which are smaller than \(N\) and coprime with \(N\). \(B_N\) is the set \(B_N = \{n \in X_N; g.c.d.(n,N) > 1\}\), \(\beta(p)\) is the number of the elements \(s\) in \(B_N\), that are coprime with \(p\) and such that \(ps\leq N\).
Set of generators (p,s) in TR of Images

Case 3: \( r=1 \) when \( N=2q \). The set of generators contains can be defined as

\[
J_{2q,2q} = \bigcup_{p_2=0}^{2q-1} (1, p_2) \cup \bigcup_{\text{g.c.d.}(p_1, 2q) > 1, p_1=0} (p_1, 1) \cup \{(2, q), (q, 2)\}.
\]

To calculate the \( N \times N \)-point discrete Fourier transform, it is sufficient to fulfill

\[
c(2q) = 2q + (q + 1) + 2 = 3(q + 1)
\]

2q-point 1D DFTs of splitting-signals \( \{f_{p,s,t}; t=0:(2q-1)\} \) generated by the set \( J_{2q,2q} \)

**Example 2:** \( q = 131 \). The \( 262 \times 262 \)-point DFT uses 396 262-point DFTs, and the column-row method uses 524 such 1-D DFTs. The tensor representation allows for reducing the number of the 1-D DFTs by 524−396=128.

<table>
<thead>
<tr>
<th>( q )</th>
<th>( 10 \times 10 )-point DFT</th>
<th>18 (versus 20) 10-point DFTs</th>
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<tbody>
<tr>
<td>5</td>
<td>14 \times 14 )-point DFT</td>
<td>24 (versus 28) 14-point DFTs</td>
</tr>
<tr>
<td>7</td>
<td>18 \times 18 )-point DFT</td>
<td>30 (versus 36) 18-point DFTs</td>
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<td>9</td>
<td>26 \times 26 )-point DFT</td>
<td>36 (versus 52) 26-point DFTs</td>
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<td>13</td>
<td>34 \times 34 )-point DFT</td>
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<tr>
<td>17</td>
<td>42 \times 42 )-point DFT</td>
<td>66 (versus 84) 42-point DFTs</td>
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The 1-D $q2^r$-point DFT

- We consider the paired algorithm for computing the $N$-point DFT,

$$F_p = (\mathcal{F}_N \circ f)_p = \sum_{n=0}^{N-1} f_n W^{np} = \sum_{t=0}^{N-1} f_{p,t} W^t, \quad p = 0:(N-1),$$

$$f_{p,t} = \sum_{n} \{f_n; \overline{np} = t\} = \sum_{n} \{f_n; np = t \mod N\}.$$ 

**Paired representation ($L>1$ a factor of $N$):**

$$p : \{f_n; n = 0:(N-1)\} \rightarrow \{f'_p,t; t = 0:(N/L-1)\}$$

$$f'_{p,t} = f'_{p,t:L} = \chi'_{p,t:L} \circ f = \sum_{k=0}^{L-1} f_{p,t+kN/L} W_L^k,$$

$$F_{(Lm+1)p} = \sum_{t=0}^{N/L-1} (f'_{p,t} W^t) W_{N/L}^{mt}, \quad m = 0:(N/L-1).$$

$$T' = T'_p = T'_{p,L} = \{(Lm+1)p \mod N; m = 0:(N/L-1)\}.$$ 

We need compose a partition $\sigma'_N = (T')$ of the set $X_N = \{0,1,..,N-1\}$ to obtain a splitting of the $N$-point DFT by small 1-D DFTs over the splitting signals.
The 1-D $q2^r$-point DFT

- The following partition of the set of all frequency-points takes place

$$\sigma'_N = (T'_{1;2}, T'_{2;2}, T'_{4;2}, T'_{8;2}, \ldots, T'_{2^r;2})$$

The N-point DFT can be reduced to transforms \( \{F_{N/2}, F_{N/4}, \ldots, F_{N/2^r}, F_q\} \),

$$[\mathcal{F}_N] = \left( \bigoplus_{n=0}^{r-1} [\mathcal{F}_{L_n}] \oplus [\mathcal{F}_q] \right) [\overline{W}][\chi'_N], \quad L_n = N/2^{n+1}$$

$$[\overline{W}] = \bigoplus_{n=0}^{r} \text{diag} \left\{ 1, W_{2L_n}^1, W_{2L_n}^2, \ldots, W_{2L_n}^{L_n-1} \right\}, \quad L_r = q.$$ 

The number of multiplications required to compute the N-point DFT is less than

$$M_{q2^r} = 2^r (M_q - 1) + q(r - 1)2^{r-1} + 2q,$$

where \( M_q \) stands for the number of multiplications in the \( q \)-point DFT.
Example (q=3, r=2): The 1-D 12-point DFT

\[ W_6^1 = \frac{1}{2} - i\frac{\sqrt{3}}{2} \]
\[ W_6^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \]

- The matrix of the 12-point DFT

\[ \mathcal{F}_{12} = \left( \bigoplus_{1}^{4} \mathcal{F}_3 \right) [\chi_{12}^2][W^3][\chi_{12}'] \]
Example \((q=3, r=2)\): The 1-D 12-point DFT

- The matrix of the 12-point DFT

\[
[F_{12}] = \left( \bigoplus_{n=1}^{3} [F_4] \bigoplus_{1}^{3} \text{diag} \{1, W_{12}^1, W_{12}^2, i\}\right)[\chi_{12;3}]
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & W & 0 & 0 & 0 & W^2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & W & 0 & 0 & 0 & W^2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & W & 0 & 0 & 0 & W^2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & W & 0 & 0 & 0 & W^2 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & W^2 & 0 & 0 & 0 & W^2 & 0 & 0 & 0 & W^2 \\
0 & 0 & W & 0 & 0 & 0 & W & 0 & 0 & 0 & W & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & W^2 & 0 & 0 & 0 & W & 0 & 0 & 0 \\
0 & W & 0 & 0 & 0 & W^2 & 0 & 0 & 0 & W & 0 & 0 \\
0 & 0 & W^2 & 0 & 0 & 0 & W & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & W^2 & 0 & 0 & 0 & W
\end{bmatrix}
\]

\[
W = W_3 = -1/2 + i\sqrt{3}/2 \\
W^2 = -1/2 - i\sqrt{3}/2
\]
Complexity: Multiplications for the 2-D DFT

The number of operations of multiplication for $q^{2^r} \times q^{2^r}$-point DFT can be estimated as

- 1. The column-row method:

$$M'_{q^{2^r}, q^{2^r}} = 2(q^{2^r} \times M'_{q^{2^r}}$$

- 2. The tensor transform-based method:

$$M'_{q^{2^r}, q^{2^r}} = c(q^{2^r}) \times M'_{q^{2^r}} \quad c(q^{2^r}) < q^{2^r+1}.$$

The number of operations of multiplications for the $q^{2^r}$-point DFT:

$$M'_{2^r} = 2^{r-1}(r-3)+2 \quad M'_{2^r3} \leq 2^{r-1}(3r - 3) + 6$$
$$M'_{2^r5} \leq 2^{r-1}(5r + 13) + 10 \quad M'_{2^r7} \leq 2^{r-1}(7r + 23) + 14.$$
Conclusion

We presented the concept of partitions revealing transforms for computing the 2-D DFT of order $q^{2r} \times q^{2r}$, where $r > 1$ and $q$ is odd number greater than 1.

- When the 2-D $q^{2r} \times q^{2r}$-point DFT is calculated by the column-row method with $2(q^{2r})$ 1-D DFTs, the fast algorithms of the 1-D DFTs of order $q^{2r}$ are required. We propose the fast algorithms which splits each 1-D DFT by the short transforms by using the fast 1-D paired transforms.

- The 2-D $q^{2r} \times q^{2r}$-point DFT can also be calculated by using the tensor or paired representations of the image, when the image is represented as a set of 1-D signals which define the 2-D transform in the different subsets of frequency-points and they all together cover the complete set of frequencies.
References

- K. Li, W. Zheng, and K. Li, “A fast algorithm with less operations for length-


THANK YOU VERY MUCH!

QUESTIONS, PLEASE?